Existence and properties of pseudo-inverses for Bessel and related processes

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Abstract It is shown that the tail probability of a Bessel process is the distributio function of a random time which is related to first and last passage times of Bessel processes.

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1 Introduction. Definition of a pseudo-inverse

1.0 The aim of this paper is to give some mathematical meaning to the assertion :
"the stochastic process $(R_t,\ t \geq 0)$, with values in $\mathbb{R}_+$, has a tendency to increase" (1.0)
and, more precisely, to "measure this tendency" when $(R_t,\ t \geq 0)$ is a Bessel process, by showing the existence of a "pseudo-inverse" of R and by studying its properties.
First, here are several possible interpretations of (1.0) :
(1.a) $(R_t,\ t \geq 0)$ is a.s. increasing ; then, it admits an increasing inverse process.
(1.b) $(R_t,\ t \geq 0)$ is a submartingale ; under some adequate conditions, it admits a Doob-Meyer decomposition :

$$R_t = M_t + A_t \qquad (t \geq 0)$$

with $(A_t,\ t \geq 0)$ an increasing process and $(M_t,\ t \geq 0)$ a martingale. Since a martingale is a "well balanced" process, $(R_t,\ t \geq 0)$ has a tendency to increase, which is measured by $(A_t,\ t \geq 0)$.

(1.c) $\big(E(R_t),\ t \geq 0\big)$ is an increasing function.
(1.d) $(R_t,\ t \geq 0)$ is "stochastically increasing", i.e. : for every $y \geq 0$, $P(R_t \geq y)$ is an increasing function of t (in the case $R_0 = x > 0$, we need to modify the previous assertion by taking $y \geq x$, to get a meaningful assertion).
We note that, trivially : (1.a) $\Longrightarrow$ (1.b) $\Longrightarrow$ (1.c) $\Longleftarrow$ (1.d) (when $x = 0$).
It is the assertion (1.d) which we have in mind throughout this study. It leads us to the definition of a pseudo-inverse of $(R_t,\ t \geq 0)$, which we now present.

1.1 Let $\big(\Omega, (R_t,\ t \geq 0), P_x(x \in \mathbb{R}_+)\big)$ denote a process taking values on $\mathbb{R}_+$, which is a.s. continuous and such that $P_x(R_0 = x) = 1$.

Definition 1.1 $(R_t,\ t \geq 0)$ *admits an increasing pseudo-inverse (resp. decreasing pseudo-inverse) if :*
i) for every $y > x$, $\lim\limits_{t \to \infty} P_x(R_t \geq y) = 1$ (1.1)
ii) for every $y > x$, *the application from* $\mathbb{R}_+$ *into* $[0, 1]$: $t \to P_x(R_t \geq y)$ *is increasing* (1.2)
$\big($*resp. if :*
i') for every $y < x$, $\lim\limits_{t \to \infty} P_x(R_t \leq y) = 1$ (1.1′)
ii') for every $y < x$, *the application from* $\mathbb{R}_+$ *into* $[0, 1]$: $t \to P_x(R_t \leq y)$ *is increasing* (1.2′)$\big)$

Definition 1.2 *Assume that (1.1) and (1.2)* $\big($*resp. (1.1') and (1.2')*$\big)$ *are satisfied. Then, there exists a family of positive r.v.'s* $(Y_{x,y},\ y > x)$ *(resp.* $Y_{x,y},\ x > y$*) such that :*

$$P_x(R_t \geq y) = P(Y_{x,y} \leq t) \qquad (t \geq 0) \tag{1.3}$$

resp.
$$P_x(R_t \leq y) = P(Y_{x,y} \leq t) \qquad (t \geq 0) \tag{1.3′}$$

We call the family of positive r.v.'s $(Y_{x,y},\ y > x)$ $\big($*resp.* $(Y_{x,y},\ y < x)\big)$ *the increasing (resp. decreasing) pseudo-inverse of the process* $(R_t,\ t \geq 0)$.
We note that, a priori, the family $(Y_{x,y},\ y > x)$ $\big($resp. $(Y_{x,y},\ y < x)\big)$ is only a family of r.v.'s, and does not constitute a process.
Note that formulae (1.3) and (1.3') may be read, either considering x as fixed and y varying or by considering the two parameters x and y as varying.

Remark 1.3
i) If $(R_t,\ t \geq 0)$ admits an increasing (resp. decreasing) pseudo-inverse, then $R_t \xrightarrow[t \to \infty]{} +\infty$ in probability $\big($resp. $R_t \xrightarrow[t \to \infty]{} 0$ in probability$\big)$.
ii) If $(R_t,\ t \geq 0)$ admits an increasing pseudo-inverse, then, for every $\alpha > 0$ (resp. for every $\alpha < 0$), the process $(R_t^\alpha,\ t \geq 0)$ admits an increasing (resp. decreasing) pseudo-inverse. If $(R_t,\ t \geq 0)$ admits a decreasing pseudo-inverse, for every $\alpha > 0$ (resp. for every $\alpha < 0$), the process $(R_t^\alpha,\ t \geq 0)$ admits a decreasing (resp. increasing) pseudo-inverse.
iii) Justifying the term "pseudo-inverse"
Condition (1.2) indicates that the process $(R_t,\ t \geq 0)$ "has a tendency to increase". In case this process is indeed increasing, we introduce $(\tau_l,\ l \geq 0)$ its right-continuous inverse :

$$\tau_l := \inf\{t \geq 0\ ;\ R_t > l\}$$

Then, from (1.3), and for $y > x$;

$$P_x(R_t \geq y) = P_x(\tau_y \leq t) = P(Y_{x,y} \leq t). \tag{1.4}$$

Thus $Y_{x,y} \overset{\text{(law)}}{=} \tau_y$ under P_x.
Hence, in this case, we may choose for the family $(Y_{x,y},\ y > x)$ the process $(\tau_y,\ y > x)$, and this justifies our terminology of pseudo-inverse.

iv) A simple example
Let $(B_t,\ t \geq 0)$ be a Brownian motion started at 0, and $a > 0$. Define : $R_t := \exp(B_t + at)$. Then, the submartingale $(R_t,\ t \geq 0)$ (which satisfies $R_0 = 1$) admits an increasing pseudo-inverse. Indeed, for every $y > 1$:

$$\begin{aligned}
P(R_t \geq y) &= P(B_t + at \geq \log y) = P(\sqrt{t}\, B_1 \geq \log y - at) \quad \text{(by scaling)} \\
&= P\left(B_1 \geq \frac{\log y}{\sqrt{t}} - a\sqrt{t}\right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\frac{\log y}{\sqrt{t}} - a\sqrt{t}}^{\infty} e^{-\frac{x^2}{2}} dx
\end{aligned}$$

Obviously, $P(R_0 \geq y) = 0$ and $\lim\limits_{t \to \infty} P(R_t \geq y) = 1$. Moreover :

$$\frac{\partial}{\partial t} P(R_t \geq y) = \frac{1}{2}\left(\frac{\log y}{\sqrt{2\pi t^3}} + \frac{a}{\sqrt{2\pi t}}\right) e^{-\frac{1}{2t}(\log y - at)^2} \geq 0$$

Thus, with $(Y_z,\ z > 0)$ defined as or, rather, satisfying :

$$P(Y_{\log y} \leq t) = P(R_t \geq y)$$

we have :

$$\begin{aligned}
P(Y_{\log y} \in dt) &= \frac{1}{2}\left(\frac{\log y}{\sqrt{2\pi t^3}} + \frac{a}{\sqrt{2\pi t}}\right) \exp\left\{-\frac{1}{2t}(\log y - at)^2\right\} dt \\
&= \frac{1}{2}\left\{P(T^{(a)}_{\log y} \in dt) + P(G^{(a)}_{\log y} \in dt)\right\}
\end{aligned}$$

with :

$$G^{(a)}_{\log y} := \sup\{t \geq 0\ ;\ B_t + at = \log y\},\ T^{(a)}_{\log y} = \inf\{t \geq 0\ ;\ B_t + at = \log y\}$$

since as is well-known (see e.g. [MRY])

$$\begin{aligned}
P(G^{(a)}_{\log y} \in dt) &= \frac{a}{\sqrt{2\pi t}} \exp\left\{-\frac{1}{2t}(\log y - at)^2\right\} dt \\
P(T^{(a)}_{\log y} \in dt) &= \frac{\log y}{\sqrt{2\pi t^3}} \exp\left\{-\frac{1}{2t}(\log y - at)^2\right\} dt \qquad (\log y,\ a > 0)
\end{aligned}$$

1.2 Pseudo-inverses and generalised Black-Scholes formulae

Our main reason to introduce this notion of a pseudo-inverse is, independently from the fact that we think it is of interest by itself, because it plays an important role in the set-up of some generalised Black-Scholes formulae. More precisely, let us denote by $(M_t,\ t \geq\ 0\ ;\ \mathcal{F}_t,\ t \geq\ 0\ ;\ P_x,\ x \geq 0)$ a Markov process taking values in $\mathbb{R}_+$, such that $(M_t,\ t \geq 0)$ is a $(\mathcal{F}_t,\ t \geq 0,\ P_x)$ positive local martingale, hence a supermartingale, which satisfies $P_x(M_0 = x) = 1$ and

$$\lim_{t\to\infty} M_t = 0 \quad P_x \qquad \text{a.s.} \tag{1.5}$$

Let $\Pi_M : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ denote the put quantity associated with $(M_t,\ t \geq 0)$, that is :

$$\Pi_M(K,t) := E_x\big((K - M_t)^+\big) \qquad (K,t \geq 0) \tag{1.6}$$

Then, it follows from our assumptions that, for every $x > 0$

i) $\left(\frac{1}{x}\,\Pi_M(y,\infty),\ 0 \leq y \leq x\right) = \left(\frac{y}{x},\ 0 \leq y \leq x\right)$ is the distribution function of a uniform r.v. U_x on $[0,x]$

ii) $\left(\frac{1}{x}\,\Pi_M(x,t),\ t \geq 0\right)$ is the distribution function of a positive r.v., since $\frac{1}{x}\,\Pi_M(x,0) = 0$, $\frac{1}{x}\,\Pi_M(x,\infty) = 1$ and the application $t \to \frac{1}{x}\,\Pi_M(x,t) = \frac{1}{x}\,E_x\big((x - M_t)^+\big)$ is an increasing function of t, $(x - M_t)^+$ being a submartingale $\big($since $(x-y)^+$ is a convex function of $y\big)$.

More precisely, it is proven in [MRY] that :

$$\frac{1}{x}\,\Pi_M(x,t) = P_x(G_x \leq t) \tag{1.6}$$

with :

$$\begin{aligned} G_x &:= \sup\{t \geq 0\ ;\ M_t = x\} \\ &\quad (= 0 \text{ if this set is empty}) \end{aligned} \tag{1.7}$$

<u>A natural question is then</u> : x being fixed, is the function of K and t $\left(\frac{1}{x}\,\Pi_M(K,t),\ K \leq\ x\ ;\ t < \infty\right)$, the distribution function of a probability on $[0,x] \times [0,\infty]$, which we denote, if it exists, by $\gamma_{M^{(x)}}$?

If so, one has

$$\frac{1}{x}\,\Pi_M(K,t) = \gamma_{M^{(x)}}\big([0,K] \times [0,t]\big) \qquad (K \leq x,\ t \geq 0) \tag{1.8}$$

On the other hand, note that, by Fubini, one has :

$$\frac{1}{x}\,\Pi_M(K,t) = \frac{1}{x}\int_0^K P_x(M_t \leq y)dy \tag{1.9}$$

Hence :

$$\frac{\partial}{\partial K}\left(\frac{1}{x}\,\Pi_M(K,t)\right) = \frac{1}{x}\,P_x(M_t \le K) \qquad (K \le x) \tag{1.10}$$

Thus, the existence of the probability $\gamma_{M^{(x)}}$ is equivalent to :

$$t \to P_x(M_t \le K) \text{ is an increasing function of } t \qquad (\text{for } K < x)$$

(for $K < x$). Clearly, this condition is equivalent to the existence of a decreasing pseudo-inverse $(Y_{x,y},\ y < x)$ for the process $(M_t,\ t \ge 0\ ;\ P_x,\ x \ge 0)$.
At this point, we have obtained :

Proposition 1.4 *For every $x > 0$, there exists a probability $\gamma_{M^{(x)}}$ on $[0,x] \times [0,\infty[$ such that :*

$$\frac{1}{x}\,\Pi_M(K,t) = \gamma_{M^{(x)}}\big([0,K] \times [0,t]\big) \qquad (K < x,\ t \ge 0) \tag{1.11}$$

if and only if $(M_t,\ t \ge 0\ ;\ P_x,\ x \ge 0)$ admits a decreasing pseudo-inverse $(Y_{x,y},\ y < x)$, i.e. if, for every $0 < y < x$ and $t \ge 0$, one has :

$$P_x(M_t \le y) = P(Y_{x,y} \le t) \tag{1.12}$$

Of course, if (1.11) is satisfied, then $\gamma_{M^{(x)}}$ is the law of a couple (U_x, Λ_x) where U_x is uniform on $[0,x]$ and $\Lambda_x \overset{(\text{law})}{=} G_x$, but U_x and Λ_x are dependent.
Moreover, if $\Pi_{M^{(x)}}$ is a regular function of K and t, then, denoting by $f_{\gamma_{M^{(x)}}}$ the density of $\gamma_{M^{(x)}}$:

$$\begin{aligned} f_{\gamma_{M^{(x)}}}(K,t) &= \frac{\partial^2}{\partial K \partial t}\,\frac{1}{x}\,\Pi_M(K,t) = \frac{1}{x}\,\frac{\partial}{\partial t}\,P_x(M_t \le y) && (1.13) \\ &\quad \big(\text{from } (1.10)\big) \\ &= \frac{1}{x}\,\frac{\partial}{\partial t}\,P(Y_{x,y} \le t) = \frac{1}{x}\,f_{Y_{x,y}}(t) \qquad (K \le x) && (1.14) \end{aligned}$$

where $f_{Y_{x,y}}$ denotes the density of $Y_{x,y}$.

1.3 Our aim in this paper

In the set-up of the classical Black-Scholes formulae, i.e. when $M_t = \mathcal{E}_t := \exp\left(B_t - \dfrac{t}{2}\right)$, where under P_x, $(B_t,\ t \ge 0)$ is a Brownian motion starting from $(\log x)$, we have proven in [MRY] the existence of a decreasing pseudo-inverse of $(M_t,\ t \ge 0)$ and thus established the existence of a probability $\gamma_{\mathcal{E}^{(x)}}$ characterized by :

$$\frac{1}{x}\,E_x\big((K - \mathcal{E}_t)^+\big) = \gamma_{\mathcal{E}^{(x)}}\big([0,K] \times [0,t]\big) \qquad (K \le x,\ t \ge 0) \tag{1.15}$$

In fact, in [MRY], the study was done for $x=1$, but reducing the study to the case $x=1$ is easy and we have also described this probability $\gamma_{\mathcal{E}(x)}$ in detail.

Now, let $\big(\Omega, (R_t, \mathcal{F}_t),\ t \geq 0\ ;\ P_x^{(\nu)},\ x \geq 0\big)$ denote a Bessel process of index ν ($\nu \geq -1$). We shall prove in Section 2 (see Theorem 2.1) the following results :

- when $\nu \geq -\dfrac{1}{2}$, the existence of an increasing pseudo-inverse $(Y_{x,y}^{(\nu)},\ y > x)$ for $(R_t,\ t \geq 0,\ P_x^{(\nu)},\ x \geq 0)$
- when $\nu = -1$, the existence of a decreasing pseudo-inverse $(Y_{x,y}^{(-1)},\ y < x)$ for $(R_t,\ t \geq\ 0,\ P_x^{(-1)},\ x \geq 0)$
- when $\nu \in \Big] -1, -\dfrac{1}{2}\Big[$:
  - *i)* if $x=0$, the existence of an increasing pseudo-inverse
  - *ii)* if $x>0$, the non-existence of a pseudo-inverse.

In Section 3, we describe the laws of the r.v.'s $(Y_{x,y}^{(\nu)},\ y > x)$. In particular, we obtain explicitely the Laplace transform of $(Y_{x,y}^{(\nu)},\ y > x)$:

$$E(e^{-\lambda Y_{x,y}^{(\nu)}}) = \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu}\,(y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \qquad \left(\nu \geq -\frac{1}{2}\right) \tag{1.16}$$

where I_ν and K_ν are the classical modified Bessel functions $\big($see [Leb], p. 108$\big)$. We also answer (partially) the question : are the r.v.'s $Y_{x,y}^{(\nu)}$ infinitely divisible ?

For $\nu > 0$, we apply (end of Section 2) the existence of a pseudo-inverse for the process $(R_t,\ t \geq 0)$ to the generalised Black-Scholes formula relative to the local martingale $(R_t^{-2\nu},\ t \geq 0)$, when $(R_t,\ t \geq 0)$ denotes a $\delta = 2(\nu+1)$ dimensional Bessel process.

In Section 4, we present two other families of Markovian positive submartingales which admit an increasing pseudo-inverse ; they are :

i) the Bessel processes with drift, studied by S. Watanabe [Wata] with infinitesimal generator :

$$L^{(\nu,a)} = \frac{1}{2}\,\frac{d^2}{dx^2} + \left(\frac{2\nu+1}{2x} + a\,\frac{I_{\nu+1}}{I_\nu}(ax)\right)\frac{d}{dx}$$

for $\nu \geq -\dfrac{1}{2}$, and $a \geq 0$;

ii) the generalized squares of Ornstein-Uhlenbeck processes $\big($see, e.g., [PY2]$\big)$, also called Cox-Ingersoll-Ross (: CIR) processes, with infinitesimal generator :

$$L^{(\nu,\beta)} = 2x\,\frac{d^2}{dx^2} + \big(2\beta x + 2(\nu+1)\big)\frac{d}{dx},$$

again, with $\nu \geq -\dfrac{1}{2}$, and $\beta \geq 0$.

Of course, letting $a \to 0$, resp. : $\beta \to 0$, in *i)*, resp. *ii)*, we recover the result for the Bessel processes.

In Section 5, we exhibit a two parameter family $(Y^{(\nu,\alpha)},\ \nu \geq 0,\ 0 \leq \alpha \leq 1)$ of variables (indexed by $(x,y),\ x<y$) which extends the family $Y^{(\nu)}$, corresponding to $\alpha = 1$.
The following paragraph aims at relating the results of Section 3, which presents the properties of the r.v.'s $Y^{(\nu)}_{x,y}$, and those of Section 5, where the r.v.'s $Y^{(\nu,\alpha)}_{x,y}$ are defined and studied.
Consider $(R^{(\nu)}_t,\ t \geq 0)$ and $(R^{(\nu')}_t,\ t \geq 0)$ two independent Bessel processes starting from 0, and with respective indices ν and ν', greater than -1, or dimensions : $d = 2(\nu+1)$ and $d' = 2(\nu'+1)$. From the additivity property of squares of Bessel processes, the process :

$$R^{(\nu+\nu'+1)}_t := \big((R^{(\nu)}_t)^2 + \big((R^{(\nu')}_t)^2\big)^{\frac{1}{2}}, \qquad t \geq 0,$$

is a Bessel process with index $\nu+\nu'+1$ (or dimension : $d+d'$), starting from 0. Let, for $y>0$:

$$\begin{aligned} G^{(\alpha)}_y &:= \sup\{t \geq 0\ :\ R^{(\alpha)}_t = y\} \\ T^{(\alpha)}_y &:= \inf\{t \geq 0\ :\ R^{(\alpha)}_t = y\} \end{aligned} \qquad (\alpha = \nu,\ \nu+\nu'+1)$$

It is then clear that :

$$T^{(\nu+\nu'+1)}_y \leq T^{(\nu)}_y \leq G^{(\nu)}_y, \qquad \text{and } G^{(\nu+\nu'+1)}_y \leq G^{(\nu)}_y.$$

These inequalities invite to look for some r.v.'s $Z^{(i)}_y$ $(i=1,2,3)$, such that :

$$G^{(\nu)}_y \overset{\text{(law)}}{=} G^{(\nu+\nu'+1)}_y + Z^{(1)}_y \tag{1.17}$$

$$G^{(\nu)}_y \overset{\text{(law)}}{=} T^{(\nu)}_y + Z^{(2)}_y \tag{1.18}$$

$$T^{(\nu)}_y \overset{\text{(law)}}{=} T^{(\nu+\nu'+1)}_y + Z^{(3)}_y \tag{1.19}$$

where the r.v.'s featured on the right-hand side of (1.17), (1.18) and (1.19) are independent.
The existence of these r.v.'s $Z^{(i)}_y$, $(i=1,2,3)$ is obtained in Sections 3 and 5 as a sub product of the existence of pseudo-inverses for Bessel processes. Precisely, the identities (1.17), (1.18) and (1.19) are shown respectively as (5.12), in Proposition 5.1, resp. as (5.8), with $y=z$ and $z=0$, resp. as (5.11) in Proposition 5.1. In Theorem 3.1 $\big($point *iv)* and *v)*$\big)$ and Theorem 5.2, other equalities in law of the type of (1.17), (1.18) and (1.19), are established, when the starting points of the Bessel processes differ from 0.

Section 6 concludes with a small list of the ingredients we have used throughout the paper to obtain these pseudo-inverses.

2 Existence of pseudo-inverses for Bessel processes

2.1 Statement of our main result

Let $\nu \geq -1$ and $(R_t,\ t \geq 0\ ;\ P_x^{(\nu)},\ x \geq 0)$ a Bessel process with index ν, i.e. with dimension $\delta = 2\nu + 2$, with $\delta \geq 0$.

Theorem 2.1

i) If $\nu \geq -\frac{1}{2}$, $(R_t,\ t \geq 0)$ admits an increasing pseudo-inverse ; that is for every $y > x > 0$, there exists a positive r.v. $Y_{x,y}^{(\nu)}$ such that, for every $t \geq 0$:

$$P_x^{(\nu)}(R_t \geq y) = P(Y_{x,y}^{(\nu)} \leq t). \tag{2.1}$$

ii) Let $\nu \in \left]-1, -\frac{1}{2}\right[$; if $x > 0$, $(R_t,\ t \geq 0)$ does not admit a pseudo-inverse. If $x = 0$, $(R_t,\ t \geq 0)$ admits an increasing pseudo-inverse (see point 2.3.2 of the proof of Theorem 2.1).

iii) If $\nu = -1$, $(R_t,\ t \geq 0)$ admits a decreasing pseudo-inverse, that is there exists, for every $x > y > 0$, a positive r.v. $Y_{x,y}^{(-1)}$ such that, for every $t \geq 0$:

$$P_x^{(-1)}(R_t \leq y) = P(Y_{x,y}^{(-1)} \leq t) \tag{2.2}$$

Thus, in Theorem 2.1, the value $\nu = -\frac{1}{2}$ appears as a critical value. This is, in fact, rather natural. Indeed, for $\nu \geq -\frac{1}{2}$, $(R_t,\ t \geq 0)$ is a submartingale - therefore it is, in a way, increasing in t - and it may be written as :

$$R_t = x + B_t + \frac{2\nu + 1}{2}\int_0^t \frac{ds}{R_s} \qquad \text{if } \nu > -\frac{1}{2}$$

where $(B_t,\ t \geq 0)$ denotes a Brownian motion starting from 0, and for $\nu = -\frac{1}{2}$, $R_t = |x + B_t| = |x| + \beta_t + L_t^{-x}$, where $(L_t^{-x},\ t \geq 0)$ denotes the local time of B at level $-x$ and $(\beta_t,\ t \geq 0)$ is a Brownian motion.

For $\nu \in \left] -1, -\frac{1}{2}\right[$, such a representation is no longer true. In fact :

$$R_t = x + \beta_t + \frac{2\nu + 1}{2}\, k_t$$

with

$$k_t := \text{p.v.} \int_0^t \frac{ds}{R_s} := \int_0^\infty a^{2\nu}(L_t^a - L_t^0)da$$

where $(L_t^a,\ t \geq 0,\ a \geq 0)$ denotes the jointly continuous family of diffusion local times associated with $(R_t,\ t \geq 0)$ $\big($see [RY], Exerc. 1.26, Chap. XI, p. 451$\big)$; in this case, $(R_t,\ t \geq 0)$ is no longer a semimartingale, but it is a Dirichlet process. $\big($See Bertoin [Ber] for a deep study of excursion theory for the process $\big((R_t, k_t),\ t \geq 0\big)\big)$.

In order to prove Theorem 2.1, we gather a few results about Bessel processes.

2.2 A summary of some results about Bessel processes

2.2.a The density of the Bessel semi-group (see [RY], Chap. XI)

We denote by $p^{(\nu)}(t,x,y)$ the density of R_t under $P_x^{(\nu)}$ $(\nu > -1)$; one has :

$$p^{(\nu)}(t,x,y) = \frac{y}{t}\left(\frac{y}{x}\right)^{\nu} \exp\left(-\frac{x^2+y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right) \qquad (\nu > -1) \tag{2.3}$$

if $x > 0$ whereas for $x = 0$

$$p^{(\nu)}(t,0,y) = 2^{-\nu} t^{-(\nu+1)}\, \Gamma(\nu+1)\, y^{2\nu+1} \exp\left(-\frac{y^2}{2t}\right) \qquad (\nu > -1) \tag{2.4}$$

where I_ν denotes the modified Bessel function with index ν (see [Leb], p. 108).

2.2.b Density of last passage times (See [PY1])

Let

$$\begin{aligned} G_y \ &:= \ \sup\{t \geq 0\ ;\ R_t = y\} \\ &\quad\ (= 0 \text{ if this set is empty}) \end{aligned} \tag{2.5}$$

Then, for $x \leq y$:

$$P_x^{(\nu)}(G_y \in dt) = \frac{\nu}{y}\, p^{(\nu)}(t,x,y)dt \qquad (\nu > 0) \tag{2.6}$$

It follows easily from (2.6) and (2.3) that, under $P_0^{(\nu)}$, the law of G_y is that of $\dfrac{y^2}{2\gamma_\nu}$, where γ_ν is a gamma r.v. with parameter ν $(\nu > 0)$:

$$G_y \overset{\text{(law)}}{=} \frac{y^2}{2\gamma_\nu} \qquad (G_y \text{ being considered under } P_0^{(\nu)}) \tag{2.7}$$

and :

$$\begin{aligned} E_x^{(\nu)}(e^{-\lambda G_y}) &= 2\nu\left(\frac{y}{x}\right)^{\nu} I_\nu(x\sqrt{2\lambda})K_\nu(y\sqrt{2\lambda}) \qquad (0 < x < y\ ;\ \nu > 0) \\ E_0^{(\nu)}(e^{-\lambda G_y}) &= \frac{1}{2^{\nu-1}\Gamma(\nu)}\,(y\sqrt{2\lambda})^{\nu} K_\nu(y\sqrt{2\lambda}) \qquad (y > 0\ ;\ \nu > 0) \end{aligned}$$

2.2.c Laplace transform of first hitting times (See [K] and [PY1])

Let

$$\begin{aligned} T_y \ &:= \ \inf\{t \geq 0\ ;\ R_t = y\} \\ &\quad\ (= +\infty \text{ if this set is empty}) \end{aligned} \tag{2.8}$$

Then :

$$E_x^{(\nu)}(e^{-\lambda T_y}) = \left(\frac{y}{x}\right)^{\nu} \frac{I_\nu(x\sqrt{2\lambda})}{I_\nu(y\sqrt{2\lambda})} \qquad (x < y,\ \lambda \geq 0) \tag{2.9}$$

In particular, for $x=0$

$$E_0^{(\nu)}(e^{-\lambda T_y}) = \frac{1}{2^\nu \Gamma(\nu+1)} \, \frac{(y\sqrt{2\lambda})^\nu}{I_\nu\,(y\sqrt{2\lambda})} \tag{2.10}$$

2.2.d Resolvent kernel (see [B.S])
Let

$$u_\lambda^{(\nu)}(x,y) = \int_0^\infty e^{-\lambda t} p^{(\nu)}(t,x,y)dt \qquad (\lambda \geq 0)$$

Then, for every positive Borel function f :

$$\int_x^\infty u_\lambda^{(\nu)}(x,y) f(y) dy = \frac{2}{x^\nu}\, I_\nu(x\sqrt{2\lambda}) \int_x^\infty y^{\nu+1} K_\nu(y\sqrt{2\lambda}) f(y) dy \tag{2.11}$$

and

$$u_\lambda^{(\nu)}(x,y) = 2y\left(\frac{y}{x}\right)^\nu I_\nu(x\sqrt{2\lambda}) K_\nu(y\sqrt{2\lambda}) \qquad (x<y)$$

where K_ν denotes the Bessel-Mc Donald function with index ν (see [Leb], p. 108).

2.2.e Bessel realisations of the Hartman-Watson laws (See [Y] and [PY1])
It was established by Hartman-Watson [H.W] that, for any $\nu \geq 0$ and $r > 0$, the ratio

$$\lambda \longrightarrow \frac{I_{\sqrt{\lambda+\nu^2}}}{I_\nu}\,(r)$$

is the Laplace transform of a probability, say $\theta_r^{(\nu)}$ on $\mathbb{R}_+$, which may then be called Hartman-Watson distribution. In [Y] and [PY1] these laws were shown to be those of the Bessel clocks $\displaystyle\int_0^t \frac{ds}{R_s^2}$ for Bessel bridges $(R_s,\ s \leq t)$ conditioned at both ends. Precisely :
For $\nu > -1$ and $\mu \neq 0$, one has :

$$E_x^{(\nu)}\left(\exp\left(-\frac{\mu^2}{2}\int_0^t \frac{ds}{R_s^2}\right)\Big| R_t = y\right) = \frac{I_{\sqrt{\mu^2+\nu^2}}}{I_\nu}\left(\frac{xy}{t}\right) \qquad (x,y \geq 0) \tag{2.12}$$

This formula may be obtained from the particular case $\nu = 0$:

$$E_x^{(0)}\left(\exp\left(-\frac{\mu^2}{2}\int_0^t \frac{ds}{R_s^2}\right)\Big| R_t = y\right) = \frac{I_{|\mu|}}{I_0}\left(\frac{xy}{t}\right)$$

together with the absolute continuity relationship :

$$P_x^{(\nu)}|_{\mathcal{R}_t \cap (t<T_0)} = \left(\frac{R_t}{x}\right)^\nu \exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{R_s^2}\right) \cdot P_x^{(0)}|_{\mathcal{R}_t} \tag{2.13}$$

where $(\mathcal{R}_t,\ t \geq 0)$ denotes the natural filtration of $(R_t,\ t \geq 0)$. If $\nu > 0$, $\mathcal{R}_t \cap (t < T_0)$ may be replaced by $\mathcal{R}_t$ in formula (2.13) since then $T_0 = \infty$ a.s., whereas if $\nu \in]-1, 0[$, then, for $\mu^2 > 0$:

$$E_x^{(\nu)} \left(\exp\left(-\frac{\mu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \Big| R_t = y \right) = E_x^{(\nu)} \left(\exp\left(-\frac{\mu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) 1_{T_0 > t} | R_t = y \right)$$

since, if $T_0 < t$, $\displaystyle\int_0^t \frac{ds}{R_s^2} = +\infty$.

We make several remarks :

i) For $\nu < 0$, letting $\mu \to 0$ in (2.12), we get :

$$P_x^{(\nu)}(T_0 > t | R_t = y) = \frac{I_{-\nu}}{I_\nu} \left(\frac{xy}{t} \right) \tag{2.14}$$

ii) For $\nu > -\frac{1}{2}$, formula (2.12) becomes, taking $\mu^2 = 2\nu + 1$:

$$E_x^{(\nu)} \left(\exp\left(-\left(\nu + \frac{1}{2}\right) \int_0^t \frac{ds}{R_s^2} \right) \Big| R_t = y \right) = \frac{I_{\nu+1}}{I_\nu} \left(\frac{xy}{t} \right) \qquad \left(\nu > -\frac{1}{2} \right) \tag{2.15}$$

From (2.15), we deduce that, for $\nu \geq -\frac{1}{2}$ and $z \geq 0$:

$$I_{\nu+1}(z) \leq I_\nu(z) \tag{2.16}$$

$\Bigg($(2.16) is an immediate consequence of (2.15) for $\nu > -\frac{1}{2}$ and for $\nu = -\frac{1}{2}$:

$I_{\frac{1}{2}}(z) = \left(\dfrac{2}{\pi z} \right)^{\frac{1}{2}} \sinh(z) \leq I_{-\frac{1}{2}}(z) = \left(\dfrac{2}{\pi z} \right)^{\frac{1}{2}} \cosh(z)$ $\big($[Leb], p. 112$\big)\Bigg)$.

2.3.f The Hirsch-Song formula $\big($see [H.S]$\big)$

Proposition 2.2 *Let $\nu > -\frac{1}{2}$.*

$$i) \qquad \frac{\partial}{\partial x} p^{(\nu)}(t, x, y) = -\frac{\partial}{\partial y} \left(\frac{x}{y} p^{(\nu+1)}(t, x, y) \right) \tag{2.17}$$

ii) Let H denote the operator defined on the space of C^1 functions, with derivative equal to 0 at 0, and which are bounded, as well as their derivative :

$$Hf(x) = \frac{1}{x} f'(x) \tag{2.18}$$

Let $(P_t^{(\nu)},\ t \geq 0)$ denotes the Bessel semi-group with index ν. Then :

$$HP_t^{(\nu)} = P_t^{(\nu+1)} H \tag{2.19}$$

iii) Let $(Q_t^{(\nu)},\ t \geq 0)$ denotes the semi-group of the Bessel squared process, with index ν. Then :

$$DQ_t^{(\nu)} = Q_t^{(\nu+1)} D \tag{2.20}$$

where D denotes the differentiation operator : $Df(x) = f'(x)$, with domain the space of C^1 functions, bounded as well as their derivative.

It is easily verified that (2.19) and (2.20) are equivalent $\big($since $Q_t^{(\nu)} f(z) = P_t^{(\nu)}(\tilde{f})(\sqrt{z})$, with $\tilde{f}(z) = f(z^2)\big)$.

On the other hand, denoting by $\overline{L}^{(\nu)}$ the infinitesimal generator of the semi-group $Q_t^{(\nu)}$: (see point 2.2.g below)

$$\overline{L}^{(\nu)} f(x) = 2x\, f''(x) + 2(\nu+1) f'(x) \tag{2.21}$$

an easy computation allows to obtain $DL^{(\nu)} = L^{(\nu+1)} D$, hence (2.20).

We show (2.17), following [HS]

For f regular with compact support, one has ;

$$P_t^{(\nu)} f(x) = \int_0^\infty p^{(\nu)}(t,x,y) f(y) dy, \text{ hence :}$$

$$\frac{\partial}{\partial x} P_t^{(\nu)} f(x) = \int_0^\infty \frac{\partial}{\partial x}\, p^{(\nu)}(t,x,y) f(y) dy \tag{2.22}$$

On the other hand, with obvious notations :

$$\frac{\partial}{\partial x}\ P_t^{(\nu)} f(x) = \frac{\partial}{\partial x}\ E^{(\nu)}\big[f(R_t^x)\big] = E^{(\nu)}\left[f'(R_t^x)\frac{\partial R_t^x}{\partial x}\right]$$

However, since :

$$R_t^x = x + B_t + \frac{2\nu+1}{2}\int_0^1 \frac{ds}{R_s^x} \qquad \left(\nu > -\frac{1}{2}\right)$$

we obtain, by differentiation with respect to x :

$$\frac{\partial R_t^x}{\partial x} = 1 - \frac{2\nu+1}{2}\int_0^t \frac{ds}{(R_s^x)^2}\ \frac{\partial R_s^x}{\partial x} \tag{2.23}$$

The linear equation (2.23) may be integrated, and we obtain :

$$\frac{\partial R_t^x}{\partial x} = \exp\left(-\frac{2\nu+1}{2}\int_0^t \frac{ds}{(R_s^x)^2}\right) \tag{2.24}$$

Thus :

$$\begin{aligned}
\int_0^\infty \frac{\partial}{\partial x}\, p^{(\nu)}(t,x,y) f(y)dy &= E_x^{(\nu)}\left[f'(R_t)\exp\left(-\frac{2\nu+1}{2}\int_0^t \frac{ds}{R_s^2}\right)\right] \\
&= \int_0^\infty E_x^{(\nu)}\left(\exp\left(-\frac{2\nu+1}{2}\int_0^t \frac{ds}{R_s^2}\right)\Big|R_t=y\right) f'(y)p^{(\nu)}(t,x,y)dy \\
&= \int_0^\infty p^{(\nu)}(t,x,y)f'(y)\frac{I_{\nu+1}}{I_\nu}\left(\frac{xy}{t}\right)dy \qquad \big(\text{from (2.15)}\big) \\
&= -\int_0^\infty f(y)\frac{\partial}{\partial y}\left(\frac{I_{\nu+1}}{I_\nu}\left(\frac{xy}{t}\right)p^{(\nu)}(t,x,y)\right)dy \\
&= -\int_0^\infty f(y)\frac{\partial}{\partial y}\left(\frac{x}{y}\, p^{(\nu+1)}(t,x,y)\right)dy \qquad \big(\text{from (2.3)}\big) \qquad (2.25)
\end{aligned}$$

The comparison of (2.25) and (2.22) then implies point *i)* of Proposition 2.2.

2.2.g The Fokker-Planck formula

The infinitesimal generator $L^{(\nu)}$ of the Bessel semi-group $(P_t^{(\nu)},\ t\geq 0)$ with index ν is given by :

$$L^{(\nu)}f(x) = \frac{1}{2}\, f''(x) + \frac{2\nu+1}{2x}\, f'(x) \qquad (2.26)$$

Its domain is the space of functions f such that $L^{(\nu)}f$ is bounded continuous and satisfies : $\lim\limits_{x\to 0} x^{2\nu+1}f'(x)=0$. One has

$$\begin{aligned}
\frac{\partial}{\partial t}\, p^{(\nu)}(t,x,y) &= L^{(\nu)}\big(p^{(\nu)}(t,\cdot,y)\big)(x) \\
&= L^{(\nu)*}\big(p^{(\nu)}(t,x,\cdot)\big)(y) \qquad (Fokker-Planck) \qquad (2.27)
\end{aligned}$$

where the operator $L^{(\nu)*}$, the adjoint of $L^{(\nu)}$, is defined by :

$$L^{(\nu)*}f(x) = \frac{1}{2}\, f''(x) - \frac{\partial}{\partial x}\left(\frac{2\nu+1}{2x}\, f(x)\right) \qquad (2.28)$$

The infinitesimal generator $\overline{L}^{(\nu)}$ of the semi-group $(Q_t^{(\nu)},\ t\geq 0)$ of the squared Bessel process with index ν, is given by :

$$\overline{L}^{(\nu)}f(x) = 2xf''(x) + (2\nu+2)f'(x) \qquad (2.29)$$

and its adjoint $\overline{L}^{(\nu)*}$ is defined by :

$$\overline{L}^{(\nu)*}f(x) = \frac{\partial^2}{\partial x^2}\big(2xf(x)\big) - (2\nu+2)f'(x) \qquad (2.30)$$

2.2.h Sojourn time below level y (see [PY1], also [MY], Th. 11.6, p. 180)
Let

$$A_y^- = \int_0^\infty 1_{(R_t \le y)} dt \left(= \int_0^{G_y} 1_{(R_t \le y)} dt \right)$$

Then :

$$E_y^{(\nu)}(e^{-\lambda A_y^-}) = \frac{2\nu}{y\sqrt{2\lambda}} \frac{I_\nu(y\sqrt{2\lambda})}{I_{\nu-1}(y\sqrt{2\lambda})} \qquad (\nu > 0 \; ; \; \lambda > 0) \tag{2.31}$$

2.3 Proof of Theorem 2.1

2.3.1 A useful Lemma

Lemma 2.3 *Let I_ν denote the modified Bessel function with index ν (see [Leb], p. 108)*

i) If $\nu \ge -\frac{1}{2}$, then, for every $z \ge 0$,

$$I_\nu(z) \ge I_{\nu+1}(z) \tag{2.32}$$

ii) If $\nu = -1$, then, for every $z \ge 0$,

$$I_1(z) = I_{-1}(z) \le I_0(z) \tag{2.33}$$

iii) If $\nu \in]-1, -\frac{1}{2}[$ for z small enough $I_\nu(z) > I_{\nu+1}(z)$ whereas for z large enough, $I_\nu(z) < I_{\nu+1}(z)$

Proof of Lemma 2.3
Point *i)* has been proven : it is relation (2.16).
We now prove point *ii)*
It follows from the previous point since $I_{-1} = I_1$.
We prove point *iii)*
In the neighborhood of 0, one has :

$$\frac{I_\nu(z)}{I_{\nu+1}(z)} \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \frac{\Gamma(\nu+2)}{(\frac{z}{2})^{\nu+1}} = \frac{2}{z} (\nu+1) \underset{z\to 0}{\longrightarrow} +\infty \qquad (\nu > -1) \tag{2.34}$$

In the neighborhood of $+\infty$, one has (see [Leb], p. 122 and 123) :

$$I_\mu(z) = \frac{e^z}{\sqrt{2\pi z}} \left(1 - \frac{4\mu^2 - 1}{4} \frac{1}{2z} + o\left(\frac{1}{z^2}\right)\right)$$

Thus, with $\mu = -\nu$ and $\nu \in \left]\frac{1}{2}, 1\right[$:

$$\begin{aligned} I_{-\nu}(z) &= \frac{e^z}{\sqrt{2\pi z}} \left(1 - \frac{4\nu^2 - 1}{4} \frac{1}{2z} + o\left(\frac{1}{z^2}\right)\right) \\ I_{-\nu+1}(z) &= \frac{e^z}{\sqrt{2\pi z}} \left(1 - \frac{4(1-\nu)^2 - 1}{4} \frac{1}{2z} + o\left(\frac{1}{z^2}\right)\right) \end{aligned}$$

Now, for $\nu \in]\frac{1}{2}, 1[$, one has $4\nu^2 - 1 \geq 4(1-\nu)^2 - 1$; indeed, this is equivalent to $\nu^2 \geq 1 + \nu^2 - 2\nu$, i.e. $\nu \geq \frac{1}{2}\cdot$

Hence, for $\nu \in]\frac{1}{2}, 1[$, the ratio $\dfrac{I_{-\nu}(z)}{I_{-\nu+1}(z)} \xrightarrow[z\to\infty]{} 1_-$ as $z \to \infty$. As a conclusion, from this point and (2.34), for $\nu \in \left]-1, -\frac{1}{2}\right[$:

- for z large enough, $\dfrac{I_\nu(z)}{I_{\nu+1}(z)} < 1$;
- for z small enough $\dfrac{I_\nu(z)}{I_{\nu+1}(z)} > 1\cdot$

2.3.2 The case $x = 0$

When $x = 0$ and $\nu > -1$, we shall show the existence of an increasing process $(J_t,\ t \geq 0)$ such that $J_t \xrightarrow[t\to\infty]{} +\infty$ a.s. and

$$P_0^{(\nu)}(R_t \geq y) = P(J_t \geq y) \tag{2.35}$$

This formula obviously implies the existence of a process which is a pseudo-inverse of $(R_t,\ t \geq 0)$. Let us prove (2.35), which hinges in fact simply on the scaling property of $(R_t,\ t \geq 0)$. Indeed, one has :

$$\begin{aligned} P_0^{(\nu)}(R_t \geq y) &= P_0^{(\nu)}\left(\sqrt{t} \geq \frac{y}{R_1}\right) \qquad \text{(by scaling)} \\ &= P_0^{(\nu)}\left(t \geq \frac{y^2}{R_1^2}\right) \\ &= P\left(t \geq \frac{y^2}{2\gamma_{\nu+1}}\right) \end{aligned}$$

$\big($since $R_1^2 \overset{\text{(law)}}{=} 2\gamma_{\nu+1}$ under $P_0^{(\nu)}$, where $\gamma_{\nu+1}$ is a gamma variable with parameter $\nu + 1\big)$.

$$\begin{aligned} &= P_0^{(\nu+1)}(G_y \leq t) \qquad \big(\text{from (2.7)}\big) \\ &= P_0^{(\nu+1)}\left(\inf_{u\geq t} R_u \geq y\right) = P_0^{(\nu+1)}(J_t \geq y) \end{aligned}$$

with $J_t := \inf_{u \geq t} R_u$. Clearly $(J_t,\ t \geq 0)$ is an increasing process and $J_t \underset{t\to\infty}{\longrightarrow} +\infty\ P_0^{(\nu+1)}$ a.s. since $\nu + 1 > 0$.

2.3.3 We now prove point *i)* of Theorem 2.1

Since, from the comparison theorem, (applied to Bessel processes) for $\nu \geq -\frac{1}{2}$, one has :

$$\begin{aligned} P_x^{(\nu)}(R_t \geq y) &\geq P_0^{(-\frac{1}{2})}(R_t \geq y) \\ &= P\big(|B_t| \geq y\big) = P_0\left(|B_1| > \frac{y}{\sqrt{t}}\right) \underset{t\to\infty}{\longrightarrow} 1 \end{aligned}$$

where $(B_t,\ t \geq 0)$ is a Brownian motion started from 0.

It suffices, in order to prove point *i)* of Theorem 2.1 to show that : $t \to P_x^{(\nu)}(R_t \geq y)$ is an increasing function of t. Now, one has :

$$\begin{aligned} \frac{\partial}{\partial t}\, P_x^{(\nu)}\, (R_t \geq y) &= \int_y^\infty \frac{\partial}{\partial t}\, p^{(\nu)}(t,x,z)dz \\ &= \int_y^\infty \left[\frac{1}{2}\,\frac{\partial^2}{\partial z^2}\, p^{(\nu)}(t,x,z) - \frac{\partial}{\partial z}\left(\frac{2\nu+1}{2z}\, p^{(\nu)}(t,x,z)\right)\right] dz \\ &\qquad \text{(from Fokker-Planck)} \\ &= -\frac{1}{2}\,\frac{\partial}{\partial y}\, p^{(\nu)}(t,x,y) + \frac{2\nu+1}{2y}\, p^{(\nu)}(t,x,y) \end{aligned}$$

Now, from (2.3) :

$$\frac{\partial}{\partial y}\, p^{(\nu)}(t,x,y) = p^{(\nu)}(t,x,y)\left[\frac{\nu+1}{y} - \frac{y}{t} + \frac{x}{t}\,\frac{I'_\nu(\frac{xy}{t})}{I_\nu(\frac{xy}{t})}\right]$$

Hence :

$$\begin{aligned} \frac{\partial}{\partial t}\, P_x^{(\nu)}(R_t \geq y) &= p^{(\nu)}(t,x,y)\left[\frac{\nu}{2y} + \frac{y}{2t} - \frac{x}{2t}\,\frac{I'_\nu(\frac{xy}{t})}{I_\nu(\frac{xy}{t})}\right] \\ &= p^{(\nu)}(t,x,y)\left[\frac{\nu}{2y} + \frac{y}{2t} - \frac{x}{2t}\left(\frac{I_{\nu+1}}{I_\nu}\left(\frac{xy}{t}\right) + \frac{\nu t}{xy}\right)\right] \\ &\left(\text{since} \qquad \frac{I'_\nu(z)}{I_\nu(z)} = \frac{I_{\nu+1}}{I_\nu}(z) + \frac{\nu}{z} \qquad \big(\text{[Leb], p. 110}\big)\right) \\ &= \frac{p^{(\nu)}(t,x,y)}{2t\, I_\nu(\frac{xy}{t})}\left[y\, I_\nu\left(\frac{xy}{t}\right) - x\, I_{\nu+1}\left(\frac{xy}{t}\right)\right] \end{aligned} \tag{2.36}$$

It now follows from Lemma 2.3 that $\frac{\partial}{\partial t}\, P_x^{(\nu)}(R_t \geq y)$ is positive, since $y > x$ and $I_\nu(z) \geq\ I_{\nu+1}(z)$.

2.3.4 We now prove point *ii)* of Theorem 2.1

This follows immediately from (2.36) and from point *iii)* of Lemma 2.3 : for $x \neq 0$, $\frac{\partial}{\partial t} P_x^{(\nu)}(R_t \geq y)$ does not have a fixed sign, hence $t \to P_x^{(\nu)}(R_t \geq y)$ cannot be a monotone function.

2.3.5 We now prove point *iii)* of Theorem 2.1

We need to show that, for every $x > 0$ and $0 < y < x$, $P_x^{(-1)}(R_t \leq y)$ is an increasing function of t. We may of course replace R_t by R_t^2. Now, the law of R_t^2, when $R_0^2 = x$, is given by :

$$P(R_t^2 \in dy) = \exp\left(-\frac{x}{2t}\right) \cdot \delta_0(dy) + q(t,x,y)dy$$

with

$$q(t,x,y) := \frac{1}{2t}\sqrt{\frac{x}{y}}\exp\left(-\frac{x+y}{2t}\right)\ I_1\left(\frac{xy}{t}\right) \tag{2.37}$$

Then, we need to show that, for every $y < x$, one has :

$$\frac{\partial}{\partial t}\left(\exp -\frac{x}{2t}\right) + \frac{\partial}{\partial t}\int_0^y q(t,x,z)dz \geq 0$$

But, since $\frac{\partial}{\partial t}\left(\exp -\frac{x}{2t}\right) \geq 0$, it suffices to show that :

$$\frac{\partial}{\partial t}\int_0^y q(t,x,z)dz \geq 0.$$

Now, one has :

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^y q(t,x,z)dz &= \int_0^y \frac{\partial}{\partial t} q(t,x,z)dz = \int_0^y \frac{\partial^2}{\partial z^2}\big(2zq(t,x,z)\big)dz \\
&\quad \text{(from Fokker-Planck)} \\
&= \frac{\partial}{\partial y}\big(2y\, q(t,x,y)\big) \\
&= 2q(t,x,y) + 2y\, q(t,x,y)\left[-\frac{1}{2y} - \frac{1}{2t} + \frac{1}{2t}\sqrt{\frac{x}{y}}\,\frac{I_1'}{I_1}\left(\frac{\sqrt{xy}}{t}\right)\right] \\
&\quad \big(\text{from (2.37)}\big) \\
&= q(t,x,y)\left[1 - \frac{y}{t} + \frac{\sqrt{xy}}{t}\left(\frac{I_0}{I_1}\left(\frac{\sqrt{xy}}{t}\right) - \frac{t}{\sqrt{xy}}\right)\right] \\
\Bigg(\text{since} \quad & \frac{I_1'}{I_1}(z) = \frac{I_0}{I_1}(z) - \frac{1}{z}\Bigg) \\
&= \frac{\sqrt{y}}{t}\, q(t,x,y)\left[\sqrt{x}\,\frac{I_0}{I_1}\left(\frac{\sqrt{xy}}{t}\right) - \sqrt{y}\right] \geq 0
\end{aligned}$$

from point *ii)* of Lemma 3.2 and since $y < x$.
This ends the proof of Theorem 2.1.

Remark 2.4
i) The property : $t \to P_x^{(\nu)}(R_t \geq y)$ is an increasing function of t $(y > x)$ is equivalent to : for every Borel increasing function φ with support in $[x, \infty[$, $t \to E_x^{(\nu)}\big[\varphi(R_t)\big]$ is an increasing function of t. Indeed, we have, for such a φ :

$$\begin{aligned} E_x^{(\nu)}\big[\varphi(R_t)\big] &= E_x^{(\nu)}\left[\int_x^{R_t} d\varphi(y)\right] = E_x^{(\nu)} \int_x^{\infty} d\varphi(y) 1_{(R_t \geq y)} \\ &= \int_x^{\infty} d\varphi(y) P_x^{(\nu)}(R_t \geq y) \end{aligned}$$

ii) Around formula (2.36)
Formula (2.36) :

$$\frac{\partial}{\partial t} P_x^{(\nu)}(R_t \geq y) = \frac{p^{(\nu)}(t,x,y)}{2t\, I_\nu(\frac{xy}{t})} \left[y\, I_\nu\left(\frac{xy}{t}\right) - x\, I_{\nu+1}\left(\frac{xy}{t}\right)\right]$$

may still be written, with the help of (2.3) :

$$\frac{\partial}{\partial t} P(Y_{x,y}^{(\nu)} \leq t) = \frac{\partial}{\partial t} P_x^{(\nu)}(R_t \geq y) = \frac{y}{2t} p^{(\nu)}(t,x,y) - \frac{x^2}{2ty} p^{(\nu+1)}(t,x,y) \tag{2.38}$$

We denote by $\theta^{(\nu)}(t,x,y)$ the RHS of (2.38). We now check directly that this positive function of t integrates to 1 i.e. : it is a density of probability :

$$\int_0^{\infty} \theta^{(\nu)}(t,x,y) dt = 1 \tag{2.39}$$

For this purpose, we use the classical Lipschitz-Hankel formula (see, e.g. [Wat], p. 384), for $\nu > 0$:

$$\nu \int_0^{\infty} \frac{du}{u} \exp(-au) I_\nu(u) = \frac{1}{(a + \sqrt{a^2-1})^\nu} \qquad (a > 1) \tag{2.40}$$

Differentiating both sides with respect to a, we obtain :

$$\int_0^{\infty} \exp(-au) \cdot I_\nu(u) du = \frac{1}{(a + \sqrt{a^2-1})^\nu} \cdot \frac{1}{\sqrt{a^2-1}} \qquad (a > 1,\ \nu > 0) \tag{2.41}$$

In order to check (2.39), we first prove that :

$$\Lambda_{x,y}^{(\nu)} := \int_0^{\infty} \frac{dt}{t} p^{(\nu)}(t,x,y) = \frac{2y}{y^2 - x^2} \qquad (y > x) \tag{2.42}$$

Note that (2.42) (which we assume for a moment) shows that $\Lambda^{(\nu)}_{x,y}$ does not depend on ν !
We deduce from (2.42) that :

$$\begin{aligned}
&\int_0^\infty \left[\frac{y}{2t}\, p^{(\nu)}(t,x,y) - \frac{x^2}{2ty}\, p^{(\nu+1)}(t,x,y)\right] dt \\
&= \frac{y}{2}\,\Lambda^{(\nu)}_{x,y} - \frac{x^2}{2y}\,\Lambda^{(\nu+1)}_{x,y} = \left(\frac{y}{2} - \frac{x^2}{2y}\right)\frac{2y}{y^2-x^2} = 1
\end{aligned}$$

which is the desired result (2.39).
<u>We now prove (2.42)</u>
From (2.3), we get :

$$\begin{aligned}
\Lambda^{(\nu)}_{x,y} &= \left(\frac{y}{x}\right)^\nu y \int_0^\infty \frac{dt}{t^2}\,\exp\left(-\frac{x^2+y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right) \\
&= \left(\frac{y}{x}\right)^\nu y \int_0^\infty du\,\exp\left(-\frac{x^2+y^2}{2}\,u\right) I_\nu(xyu) \\
&\qquad \text{(after the change of variable } \frac{1}{t} = u) \\
&= \left(\frac{y}{x}\right)^\nu \frac{y}{xy} \int_0^\infty dv\,\exp\left(-\frac{x^2+y^2}{2xy}\,v\right) I_\nu(v) \\
&\qquad \text{(after the change of variable } xyu = v) \\
&= \left(\frac{y}{x}\right)^\nu \frac{1}{x}\,\frac{1}{(a+\sqrt{a^2-1})^\nu}\,\frac{1}{\sqrt{a^2-1}}
\end{aligned}$$

with the help of (2.41), where $a := \dfrac{x^2+y^2}{2xy}$ (> 1 since ($x < y$). On the other hand, there are the elementary identities :

$$\begin{aligned}
\sqrt{a^2-1} &= \frac{y^2-x^2}{2xy},\ a+\sqrt{a^2-1} = \frac{y}{x},\ \text{which implies :} \\
&\Lambda^{(\nu)}_{x,y} = \frac{1}{x}\,\frac{1}{\frac{y^2-x^2}{2xy}} = \frac{2y}{y^2-x^2}
\end{aligned}$$

which is the desired result (2.42). We shall come back to this computation, in Section 3, in order to establish certain properties of the r.v.'s $Y^{(\nu)}_{x,y}$.

2.4 <u>Interpretation in terms of the local martingales $(R_t^{-2\nu},\ t \geq 0)$</u>

We assume here $\nu > 0$. We know that $(R_t,\ t \geq 0)$ solves the S.D.E. :

$$R_t = r + B_t + \frac{2\nu+1}{2}\int_0^t \frac{ds}{R_s} \tag{2.43}$$

where $(B_t,\ t \geq 0)$ is a Brownian motion starting from 0. Let $M_t^{(\nu)} := R_t^{-2\nu}$ and denote $M_0^{(\nu)} = x\ (= r^{-2\nu})$. Itô formula yields :

$$M_t^{(\nu)} = x - 2\nu \int_0^t (M_s^{(\nu)})^{\frac{2\nu+1}{2\nu}} dB_s. \tag{2.44}$$

We denote by $\mathbb{Q}_x^{(\nu)}$ the law of the solution of (2.44). Thus $(M_t^{(\nu)},\ t \geq 0\ ;\ \mathbb{Q}_x^{(\nu)},\ x \geq 0)$ is the family of laws of a Markov process and $(M_t^{(\nu)},\ t \geq 0)$ is a positive local martingale, hence, it is a supermartingale, which converges a.s. to 0 as $t \to \infty$. Gathering the results of Theorem 2.1, point *ii)* of Remark 1.3 and Proposition 1.4, we have obtained :

Theorem 2.4

i) For every $\nu > 0$ and $x > 0$, there exists a probability measure $\gamma_{M^{(\nu,x)}}$ on $[0,x] \times [0,\infty[$ such that :

$$\frac{1}{x}\, \mathbb{Q}_x^{(\nu)}\big((K - M_t^{(\nu)})^+\big) = \gamma_{M^{(\nu,x)}}\big([0,K] \times [0,t]\big) \qquad (K \leq x,\ t \geq 0) \tag{2.45}$$

ii) There exists a family of positive r.v.'s $(\overline{Y}_{x,y}^{(\nu)},\ y < x)$ such that :

$$\mathbb{Q}_x^{(\nu)}(M_t \leq y) = P(\overline{Y}_{x,y}^{(\nu)} \leq t) \qquad (y < x,\ t \geq 0) \tag{2.46}$$

Remark 2.5

i) Since :

$$\mathbb{Q}_x^{(\nu)}(M_t \leq y) = P^{(\nu)}_{x^{-\frac{1}{2\nu}}}\left(\frac{1}{R_t^{2\nu}} \leq y\right) = P^{(\nu)}_{x^{-\frac{1}{2\nu}}}(R_t \geq y^{-\frac{1}{2\nu}}) \tag{2.47}$$

we have :

$$\overline{Y}_{x,y}^{(\nu)} \overset{(\text{law})}{=} Y^{(\nu)}_{x^{-\frac{1}{2\nu}},y^{-\frac{1}{2\nu}}} \qquad (x > y) \tag{2.48}$$

In Section 3, we shall study in details the laws of the r.v.'s $(Y_{x,y}^{(\nu)},\ y > x)$ which, thanks to (2.48), allows to obtain easily the corresponding properties of the r.v.'s $(\overline{Y}_{x,y}^{(\nu)},\ y < x)$.

ii) When $\nu \in]-1,0[$, $(R_t^{-2\nu},\ t \geq 0)$ is no longer a local martingale, but it is a submartingale $\big($see [D.M, RVY]$\big)$ such that :

$$R_t^{-2\nu} = N_t^{(\nu)} + L_t^{(\nu)} \tag{2.49}$$

where $(N_t^{(\nu)},\ t \geq 0)$ is a martingale and $(L_t^{(\nu)},\ t \geq 0)$ an increasing process such that $dL_t^{(\nu)} = 1_{(R_t=0)} dL_t^{(\nu)}$.

3 Some properties of the r.v.'s $(Y_{x,y}^{(\nu)},\ y > x)$ $\left(\nu \geq -\frac{1}{2}\right)$

We recall that these r.v.'s $Y_{x,y}^{(\nu)}$ (or, more exactly, their laws) are defined (see Theorem 2.1) via :

$$P_x^{(\nu)}(R_t \geq y) = P(Y_{x,y}^{(\nu)} \leq t) \tag{3.1}$$

Theorem 3.1 *Here are a few properties of the laws of* $Y_{x,y}^{(\nu)}$ $\left(v \geq -\dfrac{1}{2},\ y > x\right)$

i) Laplace transform of $Y_{x,y}^{(\nu)}$

- $$\begin{aligned} E(e^{-\lambda Y_{x,y}^{(\nu)}}) &= \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu}\,(y\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(y\sqrt{2\lambda}) \qquad (x > 0,\ \lambda \geq 0) \\ &= (y\sqrt{2\lambda})\left(\frac{y}{x}\right)^\nu I_\nu(x\sqrt{2\lambda})K_{\nu+1}(y\sqrt{2\lambda}) \end{aligned} \tag{3.2}$$

For $x = 0$, we get :

- $$E(e^{-\lambda Y_{0,y}^{(\nu)}}) = \frac{1}{2^\nu\Gamma(\nu+1)}\,(y\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(y\sqrt{2\lambda}) \tag{3.3}$$

ii) Scaling property

- *The law of* $\dfrac{Y_{x,y}^{(\nu)}}{xy}$ *depends only on the ratio* $\dfrac{y}{x}$ (> 1)*. In particular :*

- $$Y_{x,y}^{(\nu)} = x^2 Y_{1,\frac{y}{x}} \qquad (y > x) \tag{3.4}$$

and

- $$E\left(e^{-\frac{\lambda Y_{x,y}^{(\nu)}}{xy}}\right) = \sqrt{2\lambda}\ b^{\nu+\frac{1}{2}} I_\nu\left(\sqrt{\frac{2\lambda}{b}}\right) K_{\nu+1}(\sqrt{2\lambda b}) \tag{3.5}$$

where $b = \dfrac{y}{x} > 1$.

iii) Further results about the laws of $Y_{x,y}^{(\nu)}$

- $$Y_{0,y}^{(\nu)} \overset{\text{(law)}}{=} \frac{y^2}{2\gamma_{\nu+1}} \overset{\text{(law)}}{=} G_y^{(\nu+1)} \tag{3.6}$$

where $G_y^{(\nu+1)}$ denotes the r.v. G_y under $P_0^{(\nu+1)}$.
The following relations hold :

- $$P(Y_{x,y}^{(\nu)} \leq t) = P\left(\gamma_{\nu+1} \geq \frac{y^2}{2t}\right) + \int_0^x \frac{z}{y}\, p^{(\nu+1)}(t,z,y)dz \tag{3.7}$$

$$\bullet \quad \frac{\partial}{\partial x}\, P(Y_{x,y}^{(\nu)} \le t) = \frac{x}{y}\, p^{(\nu+1)}(t,x,y) = \frac{x}{\nu+1}\, f_{G_{x,y}^{(\nu+1)}}(t) \tag{3.8}$$

where $f_{G_{x,y}^{(\nu+1)}}$ *denotes the density of the r.v.* G_y *under* $P_x^{(\nu+1)}$ $(x<y)$

- *The density* $f_{Y_{x,y}^{(\nu)}}$ *of* $Y_{x,y}^{(\nu)}$ *is given by :*

$$f_{Y_{x,y}^{(\nu)}}(t) = \frac{1}{2t}\left[y\, p^{(\nu)}(t,x,y) - \frac{x^2}{y}\, p^{(\nu+1)}(t,x,y)\right] \tag{3.9}$$

$$\bullet \quad E(Y_{x,y}^{(\nu)}) = \frac{y^2}{2\nu} - \frac{x^2}{2(\nu+1)} \qquad (\nu>0) \tag{3.10}$$

iv) *An equation satisfied by* $Y_{x,y}^{(\nu)}$

$$Y_{x,z}^{(\nu)} \overset{\text{(law)}}{=} T_{x,y}^{(\nu)} + Y_{y,z}^{(\nu)} \qquad (x<y<z) \tag{3.11}$$

In particular :

$$Y_{0,z}^{(\nu)} \overset{\text{(law)}}{=} T_{0,y}^{(\nu)} + Y_{y,z}^{(\nu)} \qquad (0<y<z) \tag{3.12}$$

The r.v.'s which occur on the RHS of (3.11) and (3.12) are independent and $T_{x,y}^{(\nu)}$ *is the first hitting time of level* y *by the process* $(R_t,\ t\ge 0)$ *starting from* x.

$$v) \qquad T_{0,x}^{(\nu)} + Y_{x,y}^{(\nu)} \overset{\text{(law)}}{=} G_y^{(\nu+1)} \qquad (0<x<y) \tag{3.13}$$

The r.v.'s which occur on the LHS of (3.13) are independent (*see (2.5), (2.6) and (2.7) for the definition of* $G_y^{(\nu+1)}$).

Proof of Theorem 3.1

i) We compute the Laplace transform of $Y_{x,y}^{(\nu)}$

From (3.1), we deduce :

$$\begin{aligned} \int_0^\infty e^{-\lambda t} P_x^{(\nu)}(R_t \ge y)dt &= \int_0^\infty dt \int_y^\infty e^{-\lambda t} p^{(\nu)}(t,x,z)dz \\ = \int_y^\infty dz \int_0^\infty e^{-\lambda t} p^{(\nu)}(t,x,z)dt &= \int_y^\infty u_\lambda^{(\nu)}(x,z)dz \qquad \big(\text{see (2.2.d)}\big) \\ &= E\left(\int_0^\infty e^{-\lambda t} 1_{(t\ge Y_{x,y}^{(\nu)})}dt\right) \\ &= \frac{1}{\lambda}\, E(e^{-\lambda Y_{x,y}^{(\nu)}}) \end{aligned}$$

Hence :

$$\begin{aligned} E(e^{-\lambda Y_{x,y}^{(\nu)}}) &= \lambda \int_y^\infty u_\lambda^{(\nu)}(x,z)dz \\ &= \frac{2\lambda}{x^\nu}\, I_\nu(x\sqrt{2\lambda})(2\lambda)^{-\frac{\nu}{2}-1} \int_{y\sqrt{2\lambda}}^\infty K_\nu(h)h^{\nu+1}dh \qquad \big(\text{from (2.11)}\big) \end{aligned}$$

Now, since :

$$-z^{\nu+1}K_\nu(z)=\frac{\partial}{\partial z}\big(z^{\nu+1}K_{\nu+1}(z)\big)\qquad\big(\text{[Leb], p. 110}\big)$$

we obtain :

$$\begin{aligned}E(e^{-\lambda Y_{x,y}^{(\nu)}}) &= (2\lambda)^{-\frac{1}{2}}\frac{I_\nu(x\sqrt{2\lambda})}{x^\nu}(y\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(y\sqrt{2\lambda})\\ &= \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu}(y\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(y\sqrt{2\lambda})\end{aligned}$$

Formula (3.3) may be obtained by letting x tend to 0 in (3.2) and (3.10) follows from (3.2) by differentiation.

ii) Proof of the scaling property

• It is an immediate consequence of (3.2). The fact that the law of $\dfrac{Y_{x,y}^{(\nu)}}{xy}$ depends only on the ratio $\left(\dfrac{y}{x}\right)$ $\left(\dfrac{y}{x}>1,\ x>0\right)$ may also be obtained with the help of (2.38). Indeed, we deduce from (2.36) and (2.3) that :

$$P\left(\frac{Y_{x,y}^{(\nu)}}{xy}\le t\right)=P\left(\frac{xy}{Y_{x,y}^{(\nu)}}\ge\frac{1}{t}\right)=\int_0^{txy}\frac{du}{2u}\left[y\,p^{(\nu)}(u,x,y)-\frac{x^2}{y}\,p^{(\nu+1)}(u,x,y)\right]$$

or, equivalently :

$$P\left(\frac{xy}{Y_{x,y}^{(\nu)}}\ge t\right)=\int_0^{\frac{xy}{t}}\frac{du}{2u}\left[y\,p^{(\nu)}(u,x,y)-\frac{x^2}{y}\,p^{(\nu+1)}(u,x,y)\right]$$

Given (2.3), this formula yields the density $f_{\frac{xy}{Y_{x,y}^{(\nu)}}}$ of the r.v. $\dfrac{xy}{Y_{x,y}^{(\nu)}}$:

$$f_{\frac{xy}{Y_{x,y}^{(\nu)}}}(u)=\frac{1}{2}\left(\frac{y}{x}\right)^{\nu+1}e^{-a(x,y)u}\left(I_\nu(u)-\frac{x}{y}\,I_{\nu+1}(u)\right)$$

where $a(x,y):=\dfrac{x^2+y^2}{2xy}=\dfrac{1}{2}\left(\dfrac{x}{y}+\dfrac{y}{x}\right)\cdot$

Thus, the law of $\dfrac{xy}{Y_{x,y}^{(\nu)}}$, hence that of $\dfrac{Y_{x,y}^{(\nu)}}{xy}$ only depends on the ratio $\left(\dfrac{y}{x}\right)\cdot$

• We may also prove this scaling property as a direct consequence of the scaling property of the Bessel process. Indeed, we have :

$$\begin{aligned} P(Y_{x,y}^{(\nu)} \leq t\,xy) &= P_x^{(\nu)}(R_{txy} \geq y) \\ &= P_x^{(\nu)}\left(\frac{1}{\sqrt{xy}}\,R_{txy} \geq \frac{y}{\sqrt{xy}}\right) \\ &= P_{\frac{x}{\sqrt{xy}}}^{(\nu)}\left(R_t \geq \sqrt{\frac{y}{x}}\right) \qquad \text{(by scaling of the Bessel process)} \\ &= P_{\sqrt{\frac{x}{y}}}^{(\nu)}\left(R_t \geq \sqrt{\frac{y}{x}}\right). \end{aligned}$$

iii) Proof of (3.6)
This formula (3.6) has been obtained during the proof of Theorem 2.1 (point 2.3.2, in the case $x = 0$). We note that one can deduce (3.3) from (3.6). Indeed, from (3.6), since $P(Y_{0,y}^{(\nu)} \leq t) = P\left(\frac{y^2}{2\gamma_{\nu+1}} \leq t\right)$, one has :

$$\begin{aligned} \int_0^\infty e^{-\lambda t} P(Y_{0,y}^{(\nu)} \leq t)dt &= \int_0^\infty e^{-\lambda t} P\left(\frac{y^2}{2\gamma_{\nu+1}} \leq t\right) dt, \quad \text{i.e. :} \\ E(e^{-\lambda Y_{0,y}^{(\nu)}}) &= E(e^{-\frac{\lambda}{2}\frac{y^2}{\gamma_{\nu+1}}}) \\ &= \frac{1}{\Gamma(\nu+1)} \int_0^\infty z^\nu \exp\left\{-\frac{1}{2}\left(2z + \frac{\lambda y^2}{z}\right)\right\} dz \\ &= \frac{2^{\frac{1-\nu}{2}}}{\Gamma(\nu+1)}\,\lambda^{\frac{\nu+1}{2}} y^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \qquad \big(\text{[Leb], p.119}\big) \\ &= \frac{1}{2^\nu \Gamma(\nu+1)}\,(y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda}) \end{aligned}$$

iv) We now prove (3.8) and (3.7)
Since, from the Hirsch-Song formula (2.17), one has :

$$\frac{\partial}{\partial x}\, p^{(\nu)}(t,x,y) = -\frac{\partial}{\partial y}\left(\frac{x}{y}\, p^{(\nu+1)}(t,x,y)\right)$$

then :

$$\begin{aligned} \frac{\partial}{\partial x}\, P(Y_{x,y}^{(\nu)} \leq t) &= \frac{\partial}{\partial x}\, P_x^{(\nu)}(R_t \geq y) \\ &= \int_y^\infty \frac{\partial}{\partial x}\, p^{(\nu)}(t,x,z)dz \\ &= -\int_y^\infty \frac{\partial}{\partial z}\left(\frac{x}{z}\, p^{(\nu+1)}(t,x,z)\right) dz \\ &= \frac{x}{y}\, p^{(\nu+1)}(t,x,y) \end{aligned}$$

and the relation :

$$\frac{x}{y}\, p^{(\nu+1)}(t,x,y) = \frac{x}{\nu+1}\, f_{G^{(\nu+1)}_{x,y}}(t)$$

follows from (2.6). Formula (3.7) is obtained by integration of (3.8) (with respect to x) :

$$\begin{aligned} P(Y^{(\nu)}_{x,y} \leq t) &= P(Y^{(\nu)}_{0,y} \leq t) + \int_0^x \frac{\partial}{\partial z}\, P(Y^{(\nu)}_{z,y} \leq t) dz \\ &= P\left(\gamma_{\nu+1} \geq \frac{y^2}{2t}\right) + \int_0^x \frac{z}{y}\, p^{(\nu+1)}(t,z,y) dz \end{aligned}$$

from (3.6) and (3.8). We note that (3.7) may also be written, from (3.8) :

$$P(Y^{(\nu)}_{x,y} \leq t) = P\left(\gamma_{\nu+1} \geq \frac{y^2}{2t}\right) + \int_0^x \frac{z}{\nu+1}\, f_{G^{(\nu+1)}_{x,y}}(t) dz \tag{3.14}$$

We also remark that the computation of $E(Y^{(\nu)}_{x,y})$, given by (3.10) when $\nu > 0$, may be obtained from (3.14). Indeed we deduce from (3.14) that :

$$P(Y^{(\nu)}_{x,y} \geq t) = P\left(\gamma_{\nu+1} \leq \frac{y^2}{2t}\right) - \int_0^x \frac{z}{\nu+1}\, f_{G^{(\nu+1)}_{z,y}}(t) dz$$

Thus, integrating this relation in t from 0 to ∞, for $\nu > 0$, we obtain :

$$\begin{aligned} P(Y^{(\nu)}_{x,y} \geq t) &= \int_0^\infty dt \int_0^{\frac{y^2}{2t}} \frac{1}{\Gamma(\nu+1)}\, e^{-z} z^\nu dz - \frac{x^2}{2(\nu+1)} \\ &= \frac{1}{\Gamma(\nu+1)} \int_0^\infty e^{-z} z^\nu \frac{y^2}{2z}\, dz - \frac{x^2}{2(\nu+1)} = \frac{y^2}{2\nu} - \frac{x^2}{2(\nu+1)} \end{aligned}$$

v) Formula (3.9) is an immediate consequence of (3.13).

vi) We now prove that $Y^{(\nu)}_{x,y}$ satisfies equation (3.11)

Indeed, this follows from a simple application of the Markov property. Let $x < y < z$. Since the process $(R_t,\ t \geq 0)$ starting from x needs to pass through y to reach z, we obtain :

$$\begin{aligned} P^{(\nu)}_x(R_t \geq z) &= P^{(\nu)}_x(T_y < \infty\ ;\ R_t \geq z) \\ &= P^{(\nu)}_x\big(1_{T_y \leq t}\, \widehat{P}^{(\nu)}_y(\widehat{R}_{t-T_y} \geq z)\big) \end{aligned}$$

where in the expression $\widehat{P}^{(\nu)}_y(\widehat{R}_{t-T_y} \geq z)$ the term T_y is frozen. Hence, conditioning with respect to $T_y = u$, we obtain :

$$\begin{aligned} P^{(\nu)}_x(R_t \geq z) &= \int_0^t P^{(\nu)}_x(T_y \in du) P^{(\nu)}_y(R_{t-u} \geq z) du, \quad \text{i.e. :} \\ P(Y^{(\nu)}_{x,z} \leq t) &= \int_0^t P^{(\nu)}_x(T_y \in du) P(Y^{(\nu)}_{y,z} \leq t-u) du \end{aligned}$$

hence :

$$Y^{(\nu)}_{x,z} \overset{\text{(law)}}{=} T^{(\nu)}_{x,y} + Y^{(\nu)}_{y,z}$$

It is also possible to obtain (3.11) by using the Laplace transforms of $Y^{(\nu)}_{x,y}$ and $T^{(\nu)}_{x,y}$. Indeed, from (3.2) :

$$E(e^{-\lambda Y^{(\nu)}_{x,z}}) = \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu}\,(z\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(z\sqrt{2\lambda})$$

whereas, from (3.2) and (2.9) :

$$\begin{aligned} E(e^{-\lambda(T^{(\nu)}_{x,y}+Y^{(\nu)}_{y,z})}) &= \left(\frac{y}{x}\right)^\nu \frac{I_\nu(x\sqrt{2\lambda})}{I_\nu(y\sqrt{2\lambda})}\cdot\frac{I_\nu(y\sqrt{2\lambda})}{(y\sqrt{2\lambda})^\nu}\,(z\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(z\sqrt{2\lambda}) \\ &= \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu}\,(z\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(z\sqrt{2\lambda}) = E(e^{-\lambda Y^{(\nu)}_{x,z}}) \end{aligned}$$

vii) The proof of (3.13) hinges on the same arguments as previously and, from (2.7), on :

$$E(e^{-\lambda G^{(\nu+1)}_y}) = \frac{1}{2^\nu\Gamma(\nu+1)}\,(y\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(y\sqrt{2\lambda})$$

Indeed, from (2.6), :

$$\begin{aligned} E(e^{-\lambda G^{(\nu+1)}_y}) &= E(e^{-\lambda\frac{y^2}{2\gamma_{\nu+1}}}) \\ &= \frac{1}{\Gamma(\nu+1)}\int_0^\infty e^{-\frac{\lambda y^2}{2z}}e^{-z}z^\nu dz = \frac{1}{2^\nu\Gamma(\nu+1)}\,(y\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(y\sqrt{2\lambda}) \end{aligned}$$

$\big($see [Leb], p. 119, (5.10.25)$\big)$.
This ends the proof of Theorem 3.1.

Remark 3.2
i) We note that the r.v.'s $(Y^{(\nu)}_{x,y},\ y > x)$ are not the only ones which satisfy equation (3.11). Indeed, let, for $x < y$:

$$A^{(\nu),-}_{x,y} := \int_0^\infty 1_{\left(R^{(\nu)}_x(s)\le y\right)}ds \left(= \int_0^{G_y} 1_{\left(R^{(\nu)}_x(s)\le y\right)}ds\right)$$

where here $\big(R^{(\nu)}_x(s),\ s\ge 0\big)$ denotes the Bessel process with index ν starting from x. Then, an application of the Markov property yields, for $x < y < z$:

$$A^{(\nu),-}_{x,z} \overset{\text{(law)}}{=} T^{(\nu)}_{x,y} + A^{(\nu),-}_{y,z} \tag{3.15}$$

We note that, although the r.v.'s $(A^{(\nu),-}_{x,y},\ y>x)$ satisfy the same equation as the r.v.'s $(Y^{(\nu)}_{x,y},\ y>x)$, they do not have the same law. Indeed, for $x=0$, from Ciesielski-Taylor $\big($[CT], 1962$\big)$

$$A^{(\nu),-}_{0,y} \overset{\text{(law)}}{=} T^{(\nu-1)}_{0,y} \qquad (\nu>0) \tag{3.16}$$

whereas $Y^{(\nu)}_{0,y} \overset{\text{(law)}}{=} G^{(\nu+1)}_y$ $\big($from (3.6)$\big)$ and the laws of $T^{(\nu-1)}_{0,y}$ and $G^{(\nu+1)}_y$ differ ; indeed $E(e^{-\lambda T^{(\nu-1)}_{0,y}}) = \dfrac{1}{2^{\nu-1}\Gamma(\nu)}\,\dfrac{(y\sqrt{2\lambda})^{\nu-1}}{I_{\nu-1}(y\sqrt{2\lambda})}$, from (2.10), whereas, from (3.3) and (3.6),

$$E(e^{-\lambda Y^{(\nu)}_{0,y}}) = E^{(\nu+1)}_0(e^{-\lambda G_y}) = \frac{1}{2^\nu\Gamma(\nu+1)}\,(y\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(y\sqrt{2\lambda})$$

ii) For $\nu>0$, the equality :

$$E(Y^{(\nu)}_{x,y}) = \frac{y^2}{2\nu} - \frac{x^2}{2(\nu+1)}$$

may be obtained in a different manner than the previously developed ones, by making use this time of the r.v.'s $A^{(\nu),-}_{x,y}$. Indeed, for $x<y$:

$$\begin{aligned} E(Y^{(\nu)}_{x,y}) &= \int_0^\infty P(Y^{(\nu)}_{x,y}\ge t)dt = E^{(\nu)}_x\left(\int_0^\infty 1_{(R_t\le y)}dt\right) \\ &= E(A^{(\nu),-}_{x,y}) \end{aligned} \tag{3.17}$$

Thus, we need to compute $E(A^{(\nu),-}_{x,y})$. Itô's formula, for $\nu>0$, implies, from (2.43) :

$$(R_t\wedge y)^2 = x^2 + 2\int_0^t R_s\,1_{(R_s<y)}\left(dB_s + \frac{2\nu+1}{2R_s}\,ds\right) - y\,L^y_t + \int_0^t 1_{(R_s<y)}ds \tag{3.18}$$

where $(L^y_t,\ t\ge 0)$ denotes the local time of $(R_t,\ t\ge 0)$ at level y. Hence, taking expectation in (3.18) and letting t tend to ∞, we obtain :

$$y^2 = x^2 + (2\nu+2)E(A^{(\nu),-}_{x,y}) - y\,E^{(\nu)}_x(L^y_\infty) \tag{3.19}$$

However :

$$\begin{aligned} E^{(\nu)}_x(L^y_\infty) &= E^{(\nu)}_y(L^y_\infty) = \int_0^\infty p^{(\nu)}(t,y,y)dt \\ &= \int_0^\infty \frac{y}{t}\,\exp\left(-\frac{y^2}{t}\right) I_\nu\left(\frac{y^2}{t}\right)dt \qquad \big(\text{from (2.3)}\big) \\ &= y\int_0^\infty I_\nu(z)e^{-z}\frac{dz}{z} \qquad \left(\text{after the change of variable } \frac{y^2}{t}=z\right) \\ &= \frac{y}{\nu} \end{aligned} \tag{3.20}$$

from the Lipschitz-Hankel formula $\big($see (2.40). Finally, from (3.20), (3.19) and (3.17) :

$$E(Y^{\nu}_{x,y}) = E(A^{(\nu),-}_{x,y}) = \frac{1}{2\nu+2}\left(y^2\left(1+\frac{1}{\nu}\right)-x^2\right) = \frac{y^2}{2\nu} - \frac{x^2}{2(\nu+1)}$$

iii) It may be of interest to express the law of the r.v. $Y^{(\nu)}_{x,y}$ in terms of the only process $(R_t,\ t\geq 0\ ;\ P^{(\nu)}_x)$. Here is our result :

Proposition 3.3 *Let* $\nu>0$ *and* h *a generic positive Borel function. Then :*

$$\begin{aligned} E\big(h(Y^{(\nu)}_{x,y})\big) &= \frac{y^2}{2\nu}\,E^{(\nu)}_x\left[\frac{h(G_y)}{G_y}\left(1-\frac{x}{y}\cdot\exp\left(-\left(\nu+\frac{1}{2}\right)\int_0^{G_y}\frac{ds}{R_s^2}\right)\right]\right. &(3.21)\\ &= \frac{y(y-x)}{2\nu}\,E^{(\nu)}_x\left(\frac{h(G_y)}{G_y}\right)+\frac{xy}{2\nu}\,E^{(\nu)}_x\left(\frac{h(G_y)}{G_y}\left(1-e^{-(\nu+\frac{1}{2})\int_0^{G_y}\frac{ds}{R_s^2}}\right)\right) &(3.22)\end{aligned}$$

Proof of Proposition 3.3

The density of the r.v. $Y^{(\nu)}_{x,y}$, $f_{Y^{(\nu)}_{x,y}}$, equals, from (2.36) :

$$\begin{aligned} f_{Y^{(\nu)}_{x,y}}(t) &= \frac{p^{(\nu)}(t,x,y)}{2t}\left(y-\frac{I_{\nu+1}}{I_\nu}\left(\frac{xy}{t}\right)\right)\\ &= \frac{y\,p^{(\nu)}(t,x,y)}{2t}\left(1-\frac{x}{y}\,E^{(\nu)}_x\left(\exp\left(-\left(\nu+\frac{1}{2}\right)\int_0^t\frac{ds}{R_s^2}\right)\Big|R_t=y\right)\right) \qquad (3.23)\\ &\qquad \big(\text{from (2.15)}\big)\end{aligned}$$

On the other hand, from (2.6), the density $f_{G^{(\nu)}_{x,y}}$ of G_y under $P^{(\nu)}_x$ equals :

$$f_{G^{(\nu)}_{x,y}}(t) = \frac{\nu\,p^{(\nu)}(t,x,y)}{y} \qquad (3.24)$$

Thus, since for every positive and predictable process H, one has $\big($see [FPY]$\big)$:

$$E^{(\nu)}_x(H_{G_y}) = \int_0^\infty P^{(\nu)}_x(G_y\in dt)E^{(\nu)}_x\big(H_t|R_t=y\big) \qquad (3.25)$$

we derive from (3.22) that :

$$\begin{aligned} E\big[h(Y^{(\nu)}_{x,y})\big] &= \int_0^\infty\frac{h(t)}{2t}\,p^{(\nu)}(t,x,y)\,y\left(1-\frac{x}{y}\,E^{(\nu)}_x\left(\exp\left(-\left(\nu+\frac{1}{2}\right)\int_0^t\frac{ds}{R_s^2}\right)\Big|R_t=y\right)\right.\\ &= \frac{y^2}{2\nu}\int_0^\infty\frac{h(t)}{t}\,P^{(\nu)}_x(G_y\in dt)\left(1-\frac{x}{y}\,E^{(\nu)}_x\left(\exp\left(-\left(\nu+\frac{1}{2}\right)\int_0^t\frac{ds}{R_s^2}\right)\Big|R_t=y\right)\right)\\ &= \frac{y^2}{2\nu}\,E^{(\nu)}_x\left(\frac{h(G_y)}{G_y}\left(1-\frac{x}{y}\left(\exp-\left(\nu+\frac{1}{2}\right)\int_0^{G_y}\frac{ds}{R_s^2}\right)\right)\right.\end{aligned}$$

from (3.24). This is formula (3.21), which, after some mild rearrangement, yields formula (3.22).

We note that both terms in (3.22) :

$$\frac{y(y-x)}{2\nu}\, E_x^{(\nu)}\left(\frac{h(G_y)}{G_y}\right) \quad \text{and} \quad \frac{xy}{2\nu}\, E_x^{(\nu)}\left[\frac{h(G_y)}{G_y}\big(1-e^{-(\nu+\frac{1}{2})\int_0^{G_y}\frac{ds}{R_s^2}}\big)\right]$$

are positive measures (viewed via the integration of h). The first one has total mass :

$$\begin{aligned}\frac{y(y-x)}{2\nu}\, E_x^{(\nu)}\left(\frac{1}{G_y}\right) &= \frac{y(y-x)}{2\nu}\int_0^\infty \frac{1}{ty}\,\nu\, p^{(\nu)}(t,x,y)dt \quad \big(\text{from (3.23)}\big)\\ &= \frac{y(y-x)}{2\nu}\,\frac{\nu}{y}\,\frac{2y}{y^2-x^2} = \frac{y}{y+x} \qquad \big(\text{from (2.42)}\big)\end{aligned}$$

whereas the second has total mass $\dfrac{x}{y+x}\cdot$ In particular, for $x=0$, we recover :

$$E\big[h(Y_{0,y}^{(\nu)}\big] = \frac{y^2}{2\nu}\, E_0^{(\nu)}\left(\frac{h(G_y)}{G_y}\right)$$

and the second term in (3.22) vanishes.

iv) The r.v.'s $\overline{Y}_{x,y}^{(-1)}$ $(x>y)$

In the same way that point *i)* of Theorem 2.1 allows to define the r.v.'s $\left(Y_{x,y}^{(\nu)},\ y>x\ ;\ \nu\geq -\dfrac{1}{2}\right)$, point *iii)* of this Theorem allows to define the positive r.v.'s $(\overline{Y}_{x,y}^{(-1)},\ x>y)$ characterised by :

$$Q_x^{(-1)}(M_t\leq y) = P(\overline{Y}_{x,y}^{(-1)}\leq t)$$

where $(M_t,\ t\geq 0)$ under $Q_x^{(-1)}$ is a Bessel square process with index -1 started from x. With arguments similar to those used to prove Theorem 3.1, we obtain :

$$E(e^{-\lambda\overline{Y}_{x,y}^{(-1)}}) = K_1(\sqrt{2\lambda x})\left[\sqrt{2\lambda x}+\frac{y}{\sqrt{x}}\,\sqrt{2\lambda}\, I_2(\sqrt{2\lambda y})\right] \tag{3.26}$$

v) Use of additivity property of squares of Bessel processes

We have shown, in Theorem 2.1, that for $\nu\geq -\dfrac{1}{2}$ and for $y>1$: $t\to P_1^{(\nu)}(R_t\geq y)$ is an increasing function of t. It is the distribution function of the r.v. $Y_{1,y}^{(\nu)}$ (the general case, with $x<y$, may be deduced from this one, by scaling). On the other hand, it is well known $\big($see, e.g., [RY], Chap. XI$\big)$ that $Q_x^{(d)} * Q_{x'}^{(d')} = Q_{x+x'}^{(d+d')}$, where $Q_x^{(d)}$ (resp. $Q_{x'}^{d'}$) denotes the law of a squared Bessel process, with dimension $d=2\nu+2$, starting from x (resp. with dimension $d'=2\nu'+2$ starting from x'). Hence

$$P_1^{(\nu)}(R_t\geq\sqrt{y}) = P_1^{(\nu)}(R_t^2\geq y) = P\big(R_0^2(t)+X(t)\geq y\big)$$

where $\left(R_0^2(t),\ t \geq 0\right)$ is a squared Bessel process with dimension $d = 2\nu + 2$ starting from 0 and $(X_t,\ t \geq 0)$ is a squared Bessel process with dimension 0 started from 1. Hence :

$$P_1^{(\nu)}(R_t \geq \sqrt{y}) = P\left(\gamma_{\frac{d}{2}} \geq \frac{1}{2t}(y - X_t)_+\right) \tag{3.27}$$

since $R_0^2(t) \overset{\text{(law)}}{=} t\, R_0^2(1)$ by scaling and $R_0^2(1) \overset{\text{(law)}}{=} 2\gamma_{\frac{d}{2}}$. Consequently :

$$P_1^{(\nu)}(R_t \geq \sqrt{y}) = \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty P\left(\frac{1}{2t}(y - X_t)_+ \leq z\right) z^{\frac{d}{2}-1}\, e^{-z} dz \tag{3.28}$$

Consequently, from (3.27), Theorem 2.1 hinges only upon the properties of the process $(X_t,\ t \geq 0)$, which is a squared Bessel process started from 1.
For $d = 2$ (i.e. $\nu = 0$), (3.28) becomes :

$$P_1^{(0)}(R_t \geq \sqrt{y}) = E\left(\exp -\frac{1}{2t}\,(y - X_t)_+\right) \tag{3.29}$$

since $\gamma_{\frac{d}{2}} = \gamma_1$ is then a standard exponential variable. We note that, Theorem 2.1, applied for $\nu = 0 \left(> -\frac{1}{2}!\right)$ implies that the RHS of (3.29) is a distribution function (i.e. : that of the r.v. $Y_{1,y}^{(0)}$).
For $d = 1$ $\left(\text{i.e. } \nu = -\frac{1}{2}\right)$, (3.28) becomes :

$$P_1^{(-\frac{1}{2})}(R_t \geq y) = \sqrt{\frac{2}{\pi}}\, E\left(\int_{\sqrt{\frac{(y-X_t)^+}{t}}}^\infty e^{-\frac{u^2}{2}} du\right) \tag{3.30}$$

which is also a distribution function with respect to t, that of the r.v. $Y_{1,y}^{(-\frac{1}{2})}$. This remark invites to ask the following questions :

• Which are the probabilities π on $\mathbb{R}_+$ such that : $E\left(\overline{\pi}\left(\frac{1}{2t}(y - X_t)_+\right)\right.$ is a distribution function in t, with $\overline{\pi}(x) = \pi\left(]x, \infty[\right)$? Theorem 2.1 implies that it is the case when π is a mixture of gamma laws, with parameter $\frac{d}{2} \geq \frac{1}{2}$ but that is not true if $\frac{d}{2} < \frac{1}{2}$.

• More generally, which are the properties of the process $\left(Z_t = \frac{1}{2t}(y - X_t)_+,\ t \geq 0\right)$, with $y > 1$ which may explain the above increase property ?

vi) Around infinite divisibility properties for $Y_{x,y}^{(\nu)}$

Proposition 3.4

i) Let $x > 0$ and V_x a positive, $\frac{1}{2}$ stable r.v. such that $E(e^{-\lambda V_x}) = \exp(-x\sqrt{2\lambda})$ $(\lambda \geq 0)$
If $\nu > \frac{1}{2}$, then $V_x + Y_{x,y}^{(\nu)}$ (with V_x and $Y_{x,y}^{(\nu)}$ independent) is infinitely divisible.

ii) If $\nu > \dfrac{1}{2}$ and $\dfrac{y}{x}$ is large enough, then $Y^{(\nu)}_{x,y}$ is infinitely divisible.

Proof of Proposition 3.4

We prove *i)* : By scaling we may suppose $x = 1$ and $y > 1$. From (3.2) we have :

$$\begin{aligned} E(e^{-\lambda Y^{(\nu)}_{1,y}}) &= y^{\nu+1}\sqrt{2\lambda}\, I_\nu(\sqrt{2\lambda})K_{\nu+1}(y\sqrt{2\lambda}) \\ &:= \exp\big(-h(\lambda)\big) \end{aligned} \tag{3.31}$$

Hence :

$$\begin{aligned} h'(\lambda) &= -\frac{1}{2\lambda} - \frac{1}{\sqrt{2\lambda}}\,\frac{I'_\nu(\sqrt{2\lambda})}{I_\nu(\sqrt{2\lambda})} - \frac{y}{\sqrt{2\lambda}}\,\frac{K'_{\nu+1}(y\sqrt{2\lambda})}{K_{\nu+1}(y\sqrt{2\lambda})} \\ &= \frac{y}{\sqrt{2\lambda}}\,\frac{K_\nu}{K_{\nu+1}}\,(y\sqrt{2\lambda}) - \frac{1}{\sqrt{2\lambda}}\,\frac{I_{\nu+1}}{I_\nu}\,(\sqrt{2\lambda}) \end{aligned} \tag{3.32}$$

$\Big($since $\dfrac{\partial}{\partial x}\big(x^\nu K_\nu(x) = -x^\nu K_{\nu-1}(x)$, $\dfrac{\partial}{\partial x}\big(x^{-\nu} I_\nu(x)\big) = x^{-\nu} I_{\nu+1}(x)$; see [Leb], p. 110$\Big)$. On the other hand $\big($see Ismail [I]$\big)$:

$$\frac{1}{\sqrt{2\lambda}}\,\frac{K_\nu(\sqrt{2\lambda})}{K_{\nu+1}(\sqrt{2\lambda})} = \frac{4}{\pi^2}\int_0^\infty \frac{1}{2\lambda + t^2}\,\frac{dt}{t\big(J^2_{\nu+1}(t) + Y^2_{\nu+1}(t)\big)} \tag{3.33}$$

$$\text{and : } \frac{I_{\nu+1}(\sqrt{2\lambda})}{I_\nu(\sqrt{2\lambda})} = \sqrt{2\lambda}\sum_{n=1}^\infty \frac{2}{2\lambda + j^2_{\nu,n}} \tag{3.34}$$

where $J_{\nu+1}, Y_{\nu+1}$ are the Bessel functions with index ν $\big($see [Leb], p. 98$\big)$ and $(j_{\nu,n},\ n \geq 1)$ the increasing sequence of the positive zeroes of J_ν. Hence :

$$\frac{1}{2}\,h'(\lambda) = \frac{1}{\pi}\int_0^\infty \frac{y^2}{2\lambda y^2 + t^2}\,\frac{2}{\pi t}\,\frac{dt}{J^2_{\nu+1}(t) + Y^2_{\nu+1}(t)} - \sum_{n\geq 1}\frac{1}{2\lambda + j^2_{\nu,n}} \tag{3.35}$$

Let now $Z_y := V_1 + Y^{(\nu)}_{1,y}$ $(y > 1)$. We have, from (3.35) :

$$E(e^{-\lambda Z_y}) = \exp\big(-h(\lambda) - \sqrt{2\lambda}\big) := \exp\big(-g(\lambda)\big)$$

with : $\dfrac{1}{2}\,g'(\lambda) = \dfrac{1}{2}\,h'(\lambda) + \dfrac{1}{2\sqrt{2\lambda}}$

$$\begin{aligned} &= \left[\frac{1}{\pi}\int_0^\infty \frac{y^2}{2\lambda y^2 + t^2}\,\frac{2}{\pi t}\,\frac{dt}{J^2_{\nu+1}(t) + Y^2_{\nu+1}(t)}\right] + \left[\frac{1}{\pi}\int_0^\infty \frac{dt}{2\lambda + t^2} - \sum_{n\geq 1}^\infty \frac{1}{2\lambda + j^2_{\nu,n}}\right] \\ &:= (1) + (2) \end{aligned} \tag{3.36}$$

$\left(\text{we used } \dfrac{1}{\sqrt{2\lambda}} = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{dt}{2\lambda + t^2}\right)$.

One needs, to prove *i)*, to show that g' is completely monotone. For (1), it is clear since it is the Stieltjes transform of a positive measure. For (2), one has :

$$\begin{aligned}
\frac{1}{\pi}\int_0^\infty \frac{dt}{2\lambda+t^2} - \sum_{n\geq 1}^{\infty}\frac{1}{2\lambda+j_{\nu,n}^2} &= \frac{1}{\pi}\left(\sum_{n=1}^{\infty}\int_{j_{\nu,n-1}}^{j_{\nu,n}}\frac{dt}{2\lambda+t^2} - \pi\sum_{n=1}^{\infty}\frac{1}{2\lambda+j_{\nu,n}^2}\right)\\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\int_{j_{\nu,n-1}}^{j_{\nu,n}}\left(\frac{1}{2\lambda+t^2}-\frac{1}{2\lambda+j_{\nu,n}^2}\right)dt + (j_{\nu,n}-j_{\nu,n-1}-\pi)\frac{1}{2\lambda+j_{\nu,n}^2}\right\}\\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\int_{j_{\nu,n-1}}^{j_{\nu,n}}\frac{j_{\nu,n}^2-t^2}{(2\lambda+t^2)(2\lambda+j_{\nu,n}^2)}\,dt + (j_{\nu,n}-j_{\nu,n-1}-\pi)\frac{1}{2\lambda+j_{\nu,n}^2}\right\}
\end{aligned}$$

which is clearly completely monotone since $\big($see [I], p. 357$\big)$, $j_{\nu,n}-j_{\nu,n-1}>\pi$ as $\nu>\dfrac{1}{2}\cdot$

ii) We now prove *ii)*

One needs to show that $h'(\lambda)$, as given by (3.24) is the Laplace transform, for y large enough, of a positive measure. Now, one has ;

$$\frac{1}{2}\,h'(\lambda) = \frac{1}{\pi}\int_0^\infty \frac{1}{2\lambda+v^2}\,\frac{2}{\pi v}\,\frac{dv}{J_{\nu+1}^2(yv)+Y_{\nu+1}^2(yv)} - \sum_{n\geq 1}\frac{1}{2\lambda+j_{\nu,n}^2} \tag{3.37}$$

$\big($after the change of variable $t=yv$ in (3.35)$\big)$

$$\begin{aligned}
&= \frac{1}{\pi}\left(\sum_{n=1}^{\infty}\int_{j_{\nu,n-1}}^{j_{\nu,n}}\left(\frac{1}{2\lambda+v^2}-\frac{1}{2\lambda+j_{\nu,n}^2}\right)\frac{2}{\pi v}\,\frac{dv}{J_{\nu+1}^2(yv)+Y_{\nu+1}^2(yv)}\right.\\
&\qquad \left. +\pi\sum_{n=1}^{\infty}\int_{j_{\nu,n-1}}^{j_{\nu,n}}\left(\frac{2}{\pi v}\,\frac{1}{J_{\nu+1}^2(yv)+Y_{\nu+1}^2(yv)}-1\right)dv\cdot\frac{1}{2\lambda+j_{\nu,n}^2}\right)
\end{aligned} \tag{3.38}$$

But, from Watson $\big($[Wat], p. 449$\big)$, one has :

$$J_{\nu+1}^2(z)+Y_{\nu+1}^2(z) \underset{y\to\infty}{\sim} \int_{j_{\nu,n-1}}^{j_{\nu,n}}\frac{2}{\pi v}\,\frac{\pi v y}{2}\,dv$$

Hence :

$$\begin{aligned}
\int_{j_{\nu,n-1}}^{j_{\nu,n}}\frac{2}{\pi v}\,\frac{dv}{J_{\nu+1}^2(yv)+Y_{\nu+1}^2(yv)} &\underset{y\to\infty}{\sim} \int_{j_{\nu,n-1}}^{j_{\nu,n}}\frac{2}{\pi v}\,\frac{\pi v y}{2}\,dv\\
&\underset{z\to\infty}{\sim} y(j_{\nu,n}-j_{\nu,n-1}) \qquad \text{uniformly in } n
\end{aligned}$$

Thus, for y large enough, and since $\nu>\dfrac{1}{2}$ implies $j_{\nu,n}-j_{\nu,n-1}>\pi$, one has :

$$\begin{aligned}
\frac{1}{2}\,h'(\lambda) &= \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{j_{\nu,n-1}}^{j_{\nu,n}}\frac{j_{\nu,n}^2-v^2}{(2\lambda+v^2)(2\lambda+j_{\nu,n}^2)}\,\frac{2}{\pi v}\,\frac{dv}{J_{\nu+1}^2(yv)+Y_{\nu+1}^2(yv)}\\
&\quad +\sum_{n=1}^{\infty}\alpha_n(y)\,\frac{1}{2\lambda+j_{\nu,n}^2}
\end{aligned}$$

where the $\alpha_n(y)$ are all positive. It is then clear that $h'(\lambda)$ is completely monotone.

vii) <u>On the negative moments of $Y_{x,y}^{(\nu)}$ $(x < y)$</u>
For every $m > 0$, we have :

$$E\left(\frac{1}{(Y_{x,y}^{(\nu)})^m}\right) < \infty \tag{3.39}$$

Indeed, we have, for $m > 0$:

$$E\left(\frac{1}{(Y_{x,y}^{(\nu)})^m}\right) = \frac{1}{\Gamma(m)} \int_0^\infty E(e^{-\lambda Y_{x,y}^{(\nu)}})\lambda^{m-1} d\lambda$$

and, from (3.2), $E(e^{-\lambda Y_{x,y}^{(\nu)}}) \underset{\lambda\to\infty}{\sim} \frac{1}{2}\left(\frac{y}{x}\right)^\nu \frac{1}{\sqrt{xy}}\, e^{-(y-x)\sqrt{2\lambda}}$ $\big($see [Leb], p. 123$\big)$.

4 Two extensions of Bessel processes with increasing pseudo-inverses

4.1 Bessel processes with index $\nu \geq -\frac{1}{2}$ and drift $a > 0$

S. Watanabe $\big($see, in particular [Wata], p. 117 and 118$\big)$ introduced the Bessel processes $\big((R_t,\ t \geq 0),\ P_x^{(\nu,a)},\ x \geq 0\big)$ with index ν and drift a, with extended infinitesimal generator :

$$\frac{1}{2}\,\frac{d^2}{dx^2} + \left(\frac{2\nu+1}{2x} + a\,\frac{I_{\nu+1}}{I_\nu}(ax)\right)\frac{d}{dx} \tag{4.1}$$

$\big($This is the expression (2.1) in [Wata], where we replaced α by $(2\nu+1)$, and $\sqrt{2c}$ by $a\big)$.
We recall that, for integer dimensions $d = 2(\nu+1)$, these processes may be obtained by taking $|\overrightarrow{B}_t + \overrightarrow{m}\cdot t|$, where $(\overrightarrow{B}_t)$ is a d-dimensional Brownian motion, starting from 0, and $a = |\overrightarrow{m}|$, for some $\overrightarrow{m} \in \mathbb{R}^d$.
Here is a first generalization of our Theorem 2.1 (recovered by letting $a \to 0$).

Theorem 4.1 : *For $\nu \geq -\frac{1}{2}$, and $a > 0$, the process $\big((R_t,\ t \geq 0),\ (P_x^{(\nu,a)},\ x \geq 0)\big)$ admits an increasing pseudo-inverse.*

Sketch of the proof of Theorem 4.1 (It is very similar to that of Theorem 2.1).
We need to show that, for $y > x$, $\frac{\partial}{\partial t}\, P_x^{(\nu,a)}(R_t \geq y) \geq 0$. From Fokker-Planck formula, we get :

$$\frac{\partial}{\partial t}\, p(t,x,y) = -\frac{1}{2}\,\frac{\partial}{\partial y}\, p(t,x,y) + \left(\frac{2\nu+1}{2y} + a\,\frac{I_{\nu+1}(ay)}{I_\nu(ay)}\right) p(t,x,y) \tag{4.2}$$

where $p(t,x,y)$, the density with respect to Lebesgue measure of R_t under $P_x^{(\nu,a)}$, equals, from $\big($[Wata], p. 117-118$\big)$:

$$p(t,x,y) = y\,\frac{I_\nu(ay)}{I_\nu(ax)}\,\frac{e^{-\frac{a^2t}{2}}}{t}\ \exp -\left(\frac{x^2+y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right) \tag{4.3}$$

Hence :

$$\frac{\partial}{\partial y}\,p(t,x,y) = p(t,x,y)\left[\frac{1}{y} + a\,\frac{I'_\nu(ay)}{I_\nu(ay)} - \frac{y}{t} + \frac{x}{t}\,\frac{I'_\nu(\frac{xy}{t})}{I_\nu(\frac{xy}{t})}\right]$$

Hence, we deduce from :

$$\frac{I'_\nu(z)}{I_\nu(z)} = \frac{I_{\nu+1}(z)}{I_\nu(z)} + \frac{\nu}{z}, \qquad \text{and from (4.2),}$$

that :

$$\begin{aligned}
\frac{\frac{\partial}{\partial t}\,P_x^{(\nu,a)}(R_t \geq y)}{p(t,x,y)} &= -\frac{1}{2y} - \frac{a}{2}\left(\frac{I_{\nu+1}(ay)}{I_\nu(ay)} + \frac{\nu}{ay}\right) + \frac{y}{2t} - \frac{x}{2t}\left(\frac{I_{\nu+1}(\frac{xy}{t})}{I_\nu(\frac{xy}{t})} + \frac{\nu t}{xy}\right)\\
&\quad + \frac{2\nu+1}{2y} + a\,\frac{I_{\nu+1}(ay)}{I_\nu(ay)}\\
&= \frac{a}{2}\,\frac{I_{\nu+1}(ay)}{I_\nu(ay)} + \frac{1}{2t}\left[y - x\,\frac{I_{\nu+1}(\frac{xy}{t})}{I_\nu(\frac{xy}{t})}\right] \geq 0
\end{aligned} \tag{4.4}$$

since $y \geq x$ and $I_{\nu+1}(z) \leq I_\nu(z)$, from Lemma 2.3.

4.2 Squares of generalized Ornstein-Uhlenbeck processes, also called CIR processes in mathematical finance

Let $\big((R_t,\ t\geq 0),\ (Q_x^{(\nu,\beta)},\ x \geq 0)\big)$ denote the square of a generalized Ornstein-Uhlenbeck process, with infinitesimal generator :

$$2x\,\frac{d^2}{dx^2} + \big(2\beta x + 2(\nu+1)\big)\frac{d}{dx} \tag{4.5}$$

For $d = 2(\nu+1)$ an integer, this process may be constructed as the square of the Euclidian norm of the d-dimensional Ornstein-Uhlenbeck, with parameter $\beta > 0$,that is the solution of :

$$X_t = x_0 + B_t + \beta\int_0^t X_s ds$$

where (B_t) denotes a d-dimensional Brownian motion. See, e.g., Pitman-Yor [PY2] for results about this family of diffusions.

Theorem 4.2 *For $\nu \geq -\dfrac{1}{2}$, and $\beta \geq 0$, the process $\big((R_t,\ t \geq 0)\ ;\ \big(Q_x^{(\nu,\beta)},\ x \geq 0\big)\big)$ admits an increasing pseudo-inverse.*

Sketch of the proof of Theorem 4.2 :

We need to show that $\dfrac{\partial}{\partial t}\ Q_x^{(\nu,\beta)}(R_t \geq y) \geq 0$ for every $y \geq x$. From the Fokker-Planck formula, we get :

$$\frac{\partial}{\partial t}\ Q_x^{(\nu,\beta)}(R_t \geq y) = -2y\ \frac{\partial}{\partial y}\ p(t,x,y) + (2\beta y + 2\nu)p(t,x,y) \tag{4.6}$$

where $p(t,x,y)$, the density with respect to the Lebesgue measure of R_t under $Q_x^{(\nu,\beta)}$, is given by $\big($see [PY 2]$\big)$:

$$p(t,x,y) = \frac{\beta}{2\,sinh\,\beta t}\left(\frac{y}{x}\right)^{\frac{\nu}{2}}\left\{\exp -\beta\left[(1-\nu)t + \frac{xe^{\beta t}+ye^{-\beta t}}{2\,sinh\,\beta t}\right]\right\}\cdot I_\nu\left(\frac{\beta\sqrt{xy}}{sinh\,\beta t}\right) \tag{4.7}$$

Hence :

$$\frac{\frac{\partial}{\partial y}\ p(t,x,y)}{p(t,x,y)} = \frac{\nu}{2y} - \beta\frac{e^{-\beta t}}{2\,sinh\,\beta t} + \frac{1}{2}\sqrt{\frac{x}{y}}\ \frac{\beta}{sinh\,\beta t}\ \frac{I'_\nu}{I_\nu}\left(\frac{\beta\sqrt{xy}}{sinh\,\beta t}\right) \tag{4.8}$$

Thus, from (4.8) and (4.6) :

$$\frac{\frac{\partial}{\partial t}\ Q_x^{(\nu,\beta)}(R_t \geq y)}{p(t,x,y)} = 2\beta y + \nu + \beta y\frac{e^{-\beta t}}{sinh\,\beta t} - \frac{\beta}{sinh\ \beta t}\ \sqrt{xy}\ \frac{I'_\nu}{I_\nu}\left(\frac{\beta\sqrt{xy}}{sinh\beta t}\right)$$

and, using again the relation :

$$\frac{I'_\nu}{I_\nu}(z) = \frac{I_{\nu+1}}{I_\nu}(z) + \frac{\nu}{z},$$

we have :

$$\begin{aligned}
\Delta_t := & \frac{\frac{\partial}{\partial t}\ Q_x^{(\nu,\beta)}(R_t \geq y)}{p(t,x,y)} \\
= &\ 2\beta y + \nu + \beta y\frac{e^{-\beta t}}{sinh\,\beta t} - \frac{\beta}{sinh\,\beta t}\ \sqrt{xy}\left(\frac{I_{\nu+1}}{I_\nu}\left(\frac{\beta\sqrt{xy}}{sinh\,\beta t}\right) + \frac{\nu\,sinh\,\beta t}{\beta\sqrt{xy}}\right) \\
= &\ 2\beta y + \frac{\beta y\,e^{-\beta t}}{sinh\,\beta t} - \frac{\beta}{sinh\,\beta t}\ \sqrt{xy}\ \frac{I_{\nu+1}}{I_\nu}\left(\frac{\beta\sqrt{xy}}{sinh\,\beta t}\right)
\end{aligned}$$

Denoting

$$z := \frac{\beta\sqrt{xy}}{sinh\,\beta t}, \text{ and using } y > x \Longrightarrow z \leq \frac{\beta y}{sinh\,\beta t}, \text{ we have :}$$

$$\begin{aligned}
\Delta_t &\geq 2z\,sinh\,\beta t + z\sqrt{\frac{y}{x}}\ e^{-\beta t} - z\ \frac{I_{\nu+1}}{I_\nu}(z) \quad \text{(since } y \geq x\text{)} \\
&\geq z\left(e^{\beta t} - e^{-\beta t} + e^{-\beta t} - \frac{I_{\nu+1}}{I_\nu}(z)\right) \\
&= z\left(e^{\beta t} - \frac{I_{\nu+1}}{I_\nu}(z)\right) \geq 0
\end{aligned}$$

as $\beta \geq 0$ implies $e^{\beta t} \geq 1 \geq \dfrac{I_{\nu+1}}{I_\nu}(z)$, from Lemma 2.3

Remarks : **1)** When d is an integer, the previous computation may be obtained in a simpler manner, using the fact that R_t is the square of the norm of a d-dimensional Gaussian variable with mean $x_0 \exp(\beta t)$, and (common) variance $\dfrac{e^{2\beta t}-1}{2\beta}\cdot$

2) We also deduce, from the comparison theorem for SDE's, that, for $\nu' \geq \nu$ and $\beta' \geq \beta$:

$$Q_x^{(\nu',\beta')}(R_t \geq y) \geq Q_x^{(\nu,\beta)}(R_t \geq y) \qquad (y \geq x)$$

hence, with obvious notation, for $x \leq y$, and $t \geq 0$:

$$P(Y_{x,y}^{(\nu',\beta')} \leq t) \geq P(Y_{x,y}^{(\nu,\beta)} \leq t)$$

which states that the r.v.'s. $Y_{x,y}^{(\nu,\beta)}$ are stochastically decreasing in the parameters ν and β. These variables $Y_{x,y}^{(\nu,\beta)}$ are different from those discussed in Section 5.

4.3 A third example : the process :

$$X_t^{(\nu)} = \int_0^t ds\ \exp 2\big[(B_t - B_s) + \nu(t-s)\big], \qquad t \geq 0$$

Here, $(B_t,\ t \geq 0)$ denotes a 1-dimensional Brownian motion, starting from 0. The process $(X_t^{(\nu)},\ t \geq 0)$ is easily shown to be Markov, since, thanks to Itô's formula, we obtain :

$$dX_t^{(\nu)} = \big(2(\nu+1)X_t^{(\nu)} + 1\big)dt + 2X_t^{(\nu)} d\,B_t.$$

Theorem 4.3 *For $\nu + 1 \geq 0$, the submartingale $(X_t^{(\nu)},\ t \geq 0)$ admits an increasing pseudo-inverse.*

Proof of Theorem 4.3 : Using time reversal (from t), we get : for any fixed $t \geq 0$,

$$X_t^{(\nu)} \overset{\text{(law)}}{=} A_t^{(\nu)} := \int_0^t du\ \exp\ \big(2(B_u + \nu u)\big).$$

Now, the process $(A_t^{(\nu)},\ t \geq 0)$ is increasing, and, if we denote by $(\tau_y^{(\nu)},\ y \geq 0)$ its inverse, we obtain :

$$P(X_t^{(\nu)} \geq y) = P(A_t^{(\nu)} \geq y) = P(\tau_y^{(\nu)} \leq t) \qquad (y \geq 0)$$

We note that, here, we may define the pseudo-inverse process $(Y_y^{(\nu)},\ y > 0)$ of the process $(X_t^{(\nu)},\ t \geq 0)$ as a "time" process, precisely the process $(\tau_y^{(\nu)},\ y \geq 0)$.

5 The more general family $(Y_{x,y}^{(\nu,\alpha)}\ ;\ x<y,\ \nu\geq 0,\ 0\leq\ \alpha\leq\ 1)$

In Section 2, we introduced the r.v.'s $\left(Y_{x,y}^{(\nu)},\ x<y\ ;\ \nu\geq -\frac{1}{2}\right)$ and studied their properties in Section 3. We shall now introduce further positive r.v.'s $\left(Y_{x,y}^{(\nu,\alpha)}\ ;\ x<y,\ \nu\geq 0,\ 0\leq\ \alpha\leq\ 1\right)$ which extends the family $(Y_{x,y}^{(\nu)})$, corresponding to $\alpha=1$, and we shall describe some of their properties. Let us insist again that variables have nothing to do with those introduced in Remark 2 following Theorem 4.2.

5.1 For ease of the reader, we recall some notation and formulae which we have already used and which shall be useful to us below.
$T_{x,y}^{(\nu)}$: a r.v. whose law, under $P_x^{(\nu)}$, is that of $\inf\{t\geq 0\ ;\ R_t=y\}$
$T_y^{(\nu)}=T_{0,y}^{(\nu)}$
$G_{x,y}^{(\nu)}$: a r.v. whose law, under $P_x^{(\nu)}$, is that of $\sup\{t\geq 0\ ;\ R_t=y\}$
$G_y^{(\nu)}=G_{0,y}^{(\nu)}$
Then :

$$E(e^{-\lambda T_{x,y}^{(\nu)}}) = \left(\frac{y}{x}\right)^{\nu}\frac{I_\nu(x\sqrt{2\lambda})}{I_\nu(y\sqrt{2\lambda})} \qquad (x<y) \tag{5.1}$$

$$E(e^{-\lambda T_{y}^{(\nu)}}) = \frac{1}{2^\nu\Gamma(\nu+1)}\,\frac{(y\sqrt{2\lambda})^\nu}{I_\nu(y\sqrt{2\lambda})} \qquad (y>0) \tag{5.2}$$

$$E(e^{-\lambda G_{x,y}^{(\nu)}}) = 2\nu\left(\frac{y}{x}\right)^{\nu} I_\nu(x\sqrt{2\lambda})K_\nu(y\sqrt{2\lambda}) \qquad (x<y) \tag{5.3}$$

$$E(e^{-\lambda G_{y}^{(\nu)}}) = \frac{1}{2^{\nu-1}\Gamma(\nu)}\,(y\sqrt{2\lambda})^\nu K_\nu(y\sqrt{2\lambda}) \qquad (0<y,\ \nu>0) \tag{5.4}$$

$$E(e^{-\lambda Y_{x,y}^{(\nu)}}) = \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu}\,(y\sqrt{2\lambda})^{\nu+1}K_{\nu+1}(y\sqrt{2\lambda}) \tag{5.5}$$

From these formulae, we deduced, in Theorem 3.1 :

$$T_x^{(\nu)}+Y_{x,y}^{(\nu)}\overset{\text{(law)}}{=}G_y^{(\nu+1)} \tag{5.6}$$

$$Y_{x,z}^{(\nu)}\overset{\text{(law)}}{=}T_{x,y}^{(\nu)}+Y_{x,z}^{(\nu)} \qquad (x<y<z) \tag{5.7}$$

On the other hand, it is obvious that :

$$T_{x,y}^{(\nu)}+G_{y,z}^{(\nu)}\overset{\text{(law)}}{=}G_{x,z}^{(\nu)} \qquad (x<y<z) \tag{5.8}$$

5.2 Definition of the r.v.'s $(G_y^{(\nu+\theta,\nu)}, y>0,\ \nu,\theta\geq 0)$ and $(T_y^{(\nu+\theta,\nu)}, y>0,\ \nu,\theta\geq 0)$

From Ismail-Kelker [IK], or Pitman-Yor $\big($[PY1], p. 336, formula (9.b.1),$\big)$, there exists, for every $y>0,\ \nu,\theta\geq 0$, a positive r.v. $G_y^{(\nu+\theta,\nu)}$ such that :

$$E(e^{-\lambda G_y^{(\nu+\theta,\nu)}}) = \frac{\Gamma(\nu+\theta)}{\Gamma(\nu)}\,\frac{2^\theta}{(y\sqrt{2\lambda})^\theta}\,\frac{K_\nu(y\sqrt{2\lambda})}{K_{\nu+\theta}(y\sqrt{2\lambda})} \tag{5.9}$$

Likewise from [IK], or [PY1], p. 336, formula (9.a.1), there exists, for every $y>0$, $\nu,\theta\geq 0$, a positive r.v. $T_y^{(\nu+\theta,\nu)}$ such that :

$$E(e^{-\lambda T_y^{(\nu+\theta,\nu)}}) = \frac{\Gamma(\nu+\theta+1)}{\Gamma(\nu+1)}\,\frac{2^\theta}{(y\sqrt{2\lambda})^\theta}\,\frac{I_{\nu+\theta}(y\sqrt{2\lambda})}{I_\nu(y\sqrt{2\lambda})} \tag{5.10}$$

From the relations (5.2) and (5.10) on one hand, and relations (5.4) and (5.9) on the other hand, we immediately deduce the following Proposition which completes the results of Pitman-Yor [PY1] :

Proposition 5.1

i) For every $y>0$, $\nu\geq -\dfrac{1}{2}$ and $\theta\geq 0$:

$$T_y^{(\nu+\theta)} + T_y^{(\nu+\theta,\nu)} \overset{\text{(law)}}{=} T_y^{(\nu)} \tag{5.11}$$

ii) For every $y>0$, $\nu>0$ and $\theta\geq 0$:

$$G_y^{(\nu+\theta)} + G_y^{(\nu+\theta,\nu)} \overset{\text{(law)}}{=} G_y^{(\nu)} \tag{5.12}$$

Of course, it is desirable to give a "more probabilistic" proof of (5.11) and (5.12). Here is such a proof for relation (5.11).

Another proof of (5.11)

To simplify, we take $y=1$. Let $(R_t^{(\nu)},\ t\geq 0)$ and $(R_t^{(\theta-1)},\ t\geq 0)$ two Bessel processes starting from 0, independent, with respective indices ν and $\theta-1$ (i.e. with dimension resp. $2\nu+2$ and 2θ). Let $R_t^{(\nu+\theta)} := \Big(\sqrt{(R_t^{(\nu)})^2+(R_t^{(\theta-1)})^2},\ t\geq 0\Big)\cdot (R_t^{(\nu+\theta)},\ t\geq 0)$ is a Bessel process with index $\nu+\theta$, i.e. with dimension $2\nu+2\theta+2$, started at 0. Let $T_1^{(\nu+\theta)} := \inf\{t\geq 0\ ;\ R_t^{(\nu+\theta)}=1\}$. It is clear that $T_1^{(\nu+\theta)}\leq T_1^{(\nu)}$ $\big($with $T_1^{(\nu)} := \inf\{t\geq 0\ ;\ R_t^{(\nu)}=1\}$ and that $R^{(\nu)}_{T_1^{(\nu+\theta)}}\leq 1$. Thus :

$$T_1^{(\nu)} = T_1^{(\nu+\theta)} + \inf\big\{u\geq 0\ ;\ R^{(\nu)}_{T_1^{(\nu+\theta)}+u}=1\big\} \tag{5.13}$$

On the other hand, it follows from the intertwining properties of the Bessel semi-groups $\big($see [CPY] or [DRVY]$\big)$ that :

$$E_0^{(\nu+\theta)}\big[f(R_t^{(\nu)})\big|\mathcal{R}_t^{(\nu+\alpha)}\big] = E\big[f(r\sqrt{\beta_{\nu+1,\theta}})\big] \tag{5.14}$$

with, on the right-hand side of (5.14) $r = R_t^{(\nu+\alpha)}$ and $\beta_{\nu+1,\theta}$ a beta variable with parameter $\nu+1$ and θ. We then deduce from (5.14) (which is valid for the $\mathcal{R}_t^{(\nu+\theta)}$ stopping time $T = T_1^{(\nu+\alpha)}$) that :

$$R^{(\nu)}_{T_1^{(\nu+\theta)}} \overset{\text{(law)}}{=} \sqrt{\beta_{\nu+1,\,\theta}} \tag{5.15}$$

and that $T_1^{(\nu+\theta)}$ and $(R^{(\nu)}_{T_1^{(\nu+\theta)}+u},\ u \geq 0)$ are independent. It then follows from (5.13) that :

$$E[e^{-\lambda T_1^{(\nu)}}] = E(e^{-\lambda T_1^{(\nu+\theta)}}) \cdot \int_0^1 \frac{x^\nu (1-x)^{\theta-1}}{B(\nu+1,\ \theta)}\ E(e^{-\lambda T^{(\nu)}_{\sqrt{x},1}})dx \tag{5.16}$$

Taking (5.2) into account, we shall have proven (5.11) once we establish :

$$\int_0^1 \frac{x^\nu (1-x)^{\theta-1}}{B(\nu+1,\ \theta)} \left(\frac{1}{\sqrt{x}}\right)^\nu \frac{I_\nu(\sqrt{2\lambda x})}{I_\nu(\sqrt{2\lambda})}\,dx = \frac{2^\theta \Gamma(\nu+\theta+1)}{\Gamma(\nu+1)}\ \frac{1}{(\sqrt{2\lambda})^\theta}\ \frac{I_{\nu+\theta}(\sqrt{2\lambda})}{I_\nu(\sqrt{2\lambda})} \tag{5.17}$$

Now (5.17) is easily established, by using the series expansion :

$$I_\nu(z) = \sum_{k=0}^\infty \frac{\left(\frac{Z}{2}\right)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)} \qquad \big(\text{see [Leb], p. 108}\big)$$

and then integrating term by term on the left hand side of (5.17) and using the formula :

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \qquad (a,b>0).$$

Concerning (5.12) a proof close to the preceding one seems harder since $G_1^{(\nu+\theta)}$ is not a stopping time. Nonetheless, it may be possible to use the enlarged filtration $(\mathcal{G}_t^{(\nu+\theta)},\ t \geq 0)$, i.e. the smallest filtration which contains the natural filtration $(\mathcal{R}_t^{(\nu+\theta)},\ t \geq 0)$ of $(R_t^{(\nu+\theta)},\ t \geq 0)$ and which makes $G_1^{(\nu+\theta)}$ a $(\mathcal{G}_t^{(\nu+\theta)},\ t \geq 0)$ stopping time. Explicit computations in this new filtration are made possible from the knowledge of the Azéma supermartingale $\big(P(G_1^{(\nu+\theta)} \geq t | \mathcal{R}_t^{(\nu+\theta)}),\ t \geq 0\big) = \big((R_t^{(\nu+\theta)})^{-2(\nu+\theta)} \wedge 1,\ t \geq 0\big)$. The enlargement formulae allowing to express the process $(R_t^{(\nu+\theta)},\ t \geq 0)$ in this new filtration lead to complicated formulae which do not seem to provide us with the desired explicit result.

5.3 Here is now the main result of this Section 5, i.e. the existence and the properties of the r.v.'s $\big(Y_{x,y}^{(\nu,\alpha)}\ ;\ x<y\ ;\ \nu \geq 0,\ \alpha \in [0,1]\big)$.

Theorem 5.2

i) For every $\nu \geq 0$, $x<y$ and $\alpha \in [0,1]$, there exists a positive r.v. $Y_{x,y}^{(\nu,\alpha)}$ such that :

$$E(e^{-\lambda Y_{x,y}^{(\nu,\alpha)}}) = \frac{2^{1-\alpha}\Gamma(\nu+1)}{\Gamma(\nu+\alpha)}\ \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu}\ (y\sqrt{2\lambda})^{\nu+\alpha} K_{\nu+\alpha}(y\sqrt{2\lambda}) \tag{5.18}$$

Moreover :

ii) The r.v.'s $Y_{x,y}^{(\nu,\alpha)}$ *interpolate between* $Y_{x,y}^{(\nu)}$ *and* $G_{x,y}^{(\nu)}$*, i.e. :*

$$Y_{x,y}^{(\nu,1)} \overset{\text{(law)}}{=} Y_{x,y}^{(\nu)}, \qquad Y_{x,y}^{(\nu,0)} \overset{\text{(law)}}{=} G_{x,y}^{(\nu)} \tag{5.19}$$

iii) $$Y_{x,y}^{(\nu,\alpha)} \overset{\text{(law)}}{=} Y_{x,y}^{(\nu)} + G_y^{(\nu+1,\,1-\alpha)} \tag{5.20}$$

and, for $(1-\nu)^+ \leq \alpha \leq 1$

$$Y_{x,y}^{(\nu,\alpha)} \overset{\text{(law)}}{=} Y_{x,y}^{(\nu+\alpha-1)} + T_x^{(\nu,\,1-\alpha)} \tag{5.21}$$

iv) $$T_x^{(\nu)} + Y_{x,y}^{(\nu,\alpha)} \overset{\text{(law)}}{=} G_y^{(\nu+\alpha)} \overset{\text{(law)}}{=} T_x^{(\nu+\alpha-1)} + Y_{x,y}^{(\nu+\alpha-1)} \tag{5.22}$$

v) $$G_{x,y}^{(\nu)} = G_y^{(\nu+\alpha,\,\alpha)} + Y_{x,y}^{(\nu,\alpha)} \tag{5.23}$$

In particular, for $x = 0$ *we recover (5.12) :*

$$G_y^{(\nu)} \overset{\text{(law)}}{=} G_y^{(\nu+\alpha,\,\alpha)} + G_y^{(\nu+\alpha)} \tag{5.24}$$

vi) $$Y_{x,y}^{(\nu)} + G_y^{(\nu+1,\,1-\alpha)} \overset{\text{(law)}}{=} Y_{x,y}^{(\nu+\alpha-1)} + T_x^{(\nu,1-\alpha)} \quad \big(\overset{\text{(law)}}{=} Y_{x,y}^{(\nu,\alpha)}\big) \tag{5.25}$$

Observe that, from (5.20) and (5.24), the r.v.'s $Y_{x,y}^{(\nu,\alpha)}$ are stochastically decreasing in α.

Proof of Theorem 5.2

i) We show (5.18). We have :

$$\begin{aligned}
&\frac{2^{1-\alpha}\Gamma(\nu+1)}{\Gamma(\nu+\alpha)}\ \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu}\ (y\sqrt{2\lambda})^{\nu+\alpha} K_{\nu+\alpha}(y\sqrt{2\lambda}) \\
&\quad = \left\{\frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu}\ (y\sqrt{2\lambda})^{\nu+1} K_{\nu+1}(y\sqrt{2\lambda})\right\} \\
&\qquad\quad \cdot \left\{\frac{2^{1-\alpha}\Gamma(\nu+1)}{\Gamma(\nu+\alpha)}\ (y\sqrt{2\lambda})^{\alpha-1} \frac{K_{\nu+\alpha}(y\sqrt{2\lambda})}{K_{\nu+1}(y\sqrt{2\lambda})}\right\} \\
&\quad = E(e^{-\lambda Y_{x,y}^{(\nu)}}) \cdot E(e^{-\lambda G_y^{(\nu+1,\,1-\alpha)}}) \qquad \big(\text{from (5.5) and (5.9)}\big)
\end{aligned}$$

This proves (5.18) and (5.20).

ii) We now show (5.21). We have :

$$\begin{aligned}
E(e^{-\lambda Y_{x,y}^{(\nu,\alpha)}}) &= \frac{2^{1-\alpha}\Gamma(\nu+1)}{\Gamma(\nu+\alpha)}\ \frac{I_\nu(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^\nu}\ (y\sqrt{2\lambda})^{\nu+\alpha} K_{\nu+1}(y\sqrt{2\lambda}) \\
= &\left\{\frac{I_{\nu+\alpha-1}(x\sqrt{2\lambda})}{(x\sqrt{2\lambda})^{\nu+\alpha-1}}\ (y\sqrt{2\lambda})^{\nu+\alpha} K_{\nu+\alpha}(y\sqrt{2\lambda})\right\} \\
&\quad \cdot \left\{\frac{2^{1-\alpha}\Gamma(\nu+1)}{\Gamma(\nu+\alpha)}\ \frac{1}{(x\sqrt{2\lambda})^{1-\alpha}}\ \frac{I_\nu(x\sqrt{2\lambda})}{I_{\nu+\alpha-1}(x\sqrt{2\lambda})}\right\} \\
&= E(e^{-\lambda Y_{x,y}^{(\nu+\alpha-1)}}) \cdot E(e^{-\lambda T_x^{(\nu,\,1-\alpha)}}) \qquad \big(\text{from (5.5) and (5.10)}\big)
\end{aligned}$$

This shows (5.21).

iii) The relation $T_x^{(\nu)} + Y_{x,y}^{(\nu,\alpha)} \overset{\text{(law)}}{=} G_y^{(\nu+\alpha)}$ follows immediately from (5.2), (5.18) and (5.4). The relation $G_y^{(\nu+\alpha)} \overset{\text{(law)}}{=} T_x^{(\nu+\alpha-1)} + Y_{x,y}^{(\nu+\alpha-1)}$ follows from (5.6) $\big($or from the preceding relation where we replace ν by $\nu+\alpha-1$ and we observe, as is obvious, that $Y_{x,y}^{(\nu+\alpha-1)} \overset{\text{(law)}}{=} Y_{x,y}^{(\nu+\alpha-1,\,1)}\big)$.

iv) We now show (5.23). We deduce, from (5.8), (5.24) and (5.22) :

$$\begin{aligned} T_x^{(\nu)} + G_{x,y}^{(\nu)} \overset{\text{(law)}}{=} G_y^{(\nu)} &\overset{\text{(law)}}{=} G_y^{(\nu+\alpha,\,\alpha)} + G_y^{(\nu+\alpha)} \\ &\overset{\text{(law)}}{=} G_y^{(\nu+\alpha,\,\alpha)} + T_x^{(\nu)} + Y_{x,y}^{(\nu+\alpha)} \end{aligned}$$

Hence :

$$G_{x,y}^{(\nu)} \overset{\text{(law)}}{=} G_y^{(\nu+\alpha,\,\alpha)} + Y_{x,y}^{(\nu,\alpha)}$$

We might also have proven this last relation by using (5.3), (5.9) and (5.18).

v) The relation (5.19) obviously holds.

vi) Finally, relation (5.25) follows from (5.21) and (5.20). Indeed, from (5.21) and (5.20) :

$$\begin{aligned} Y_{x,y}^{(\nu,\alpha)} &\overset{\text{(law)}}{=} Y_{x,y}^{(\nu+\alpha-1)} + T_x^{(\nu,\,1-\alpha)} \\ &\overset{\text{(law)}}{=} T_{x,y}^{(\nu)} + G_y^{(\nu+1,\,1-\alpha)} \qquad \big(x<y,\ (1-\nu)^+ \le \alpha \le 1\big) \end{aligned}$$

6 Concluding remarks : our tools to show existence of pseudo-inverses

6.1 At the beginning of this paper $\big($see point *iv)* of Remark 1.3$\big)$ we showed that the geometric Brownian motion $\big(\exp(B_t+\nu t),\ t\ge 0\big)$, with $\nu>0$, admits an increasing pseudo-inverse. On the other hand, Lamperti's representation of geometric Brownian motion is $\big($see [RY], Exerc. 1.28, Chap. XI, p. 452$\big)$:

$$\begin{aligned} \exp(B_t+\nu t) &= R^{(\nu)}\left(\int_0^t \big(\exp 2(B_s+\nu s)\big)ds\right) \qquad (6.1) \\ &= R^{(\nu)}\big(C^{(\nu)}(t)\big) \end{aligned}$$

with $C^{(\nu)}(t) := \displaystyle\int_0^t \big(\exp 2(B_s+\nu s)\big)ds$ and $\big(R^{(\nu)}(u),\ u\ge 0\big)$ a Bessel process with index ν. This formula (6.1) may induce the reader in thinking that Theorem 2.1 – the existence of an increasing pseudo-inverse for $\big(R^{(\nu)}(u),\ u\ge 0\big)$ – may be deduced from (6.1) and point *iv)* of Remark 1.3. This is not at all clear since the clock $(C^{(\nu)}(t),\ t\ge 0)$ is random.

6.2 For the proof of Theorem 2.1, we have used, essentially, among the properties of Bessel processes :

- the Fokker-Planck formula, which allowed us to compute $\dfrac{\partial}{\partial t}\, P_x^{(\nu)}(R_t \geq y)$
- the Pitman-Yor formula (2.15), which allowed to show that $\dfrac{I_{\nu+1}}{I_\nu}(z) \leq 1$, a key point to prove that $P_x^{(\nu)}(R_t \geq y)$ is increasing in t
- the formula due to Hirsch-Song, which allowed to give several expressions of the law of the r.v. $Y_{x,y}^{(\nu)}$ $\big($for $\nu \geq -\dfrac{1}{2}\big)$
- the formula for the resolvent kernel, thanks to which we were able to compute explicitely the Laplace transform of the r.v.'s $Y_{x,y}^{(\nu)}$.

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